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Random Walk Algorithms: Lecture 6

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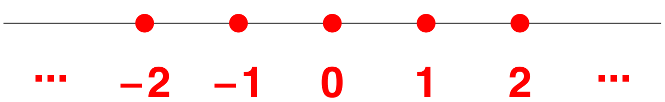
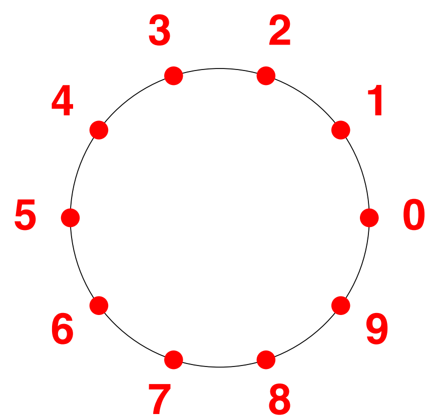
In which random walks are illustrated and evolved, and a conserved quantity is discovered, leading to consideration of a new Markov process comprised of two steps of a random walk.

Diagrams

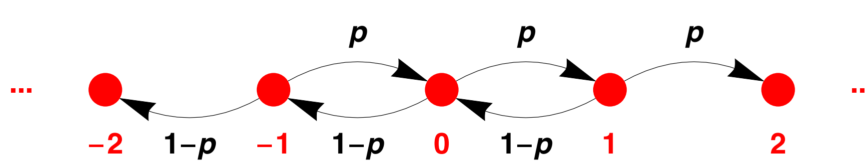
Here are the two random walks we have seen so far, on $\mathbb{Z}$ and on $\mathbb{Z}/\ell\mathbb{Z}$, with $\ell = 10$:





$$
\begin{array}{c}
 \\ \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} \ldots & -2 & -1 & 0 & 1 & 2 & \ldots \end{array} \\
\begin{pmatrix}
 & & & & & & \\
 & \ddots & 1-p & & & & \\
 & & 0 & 1-p & & & \\
 & & p & 0 & 1-p & & \\
 & & & p & 0 & & \\
 & & & & p & \ddots & \\
 & & & & & &
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c|c}
 & \begin{array}{ccccc} 0 & 1 & \ldots & 8 & 9 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \vdots \\ 8 \\ 9 \end{array} &
\begin{pmatrix}
0 & 1-p & & & p \\
p & 0 & & & \\
 & \ddots & \ddots & \ddots & \\
 & & & 0 & 1-p \\
1-p & & & p & 0
\end{pmatrix}.
\end{array}
$$

The vertices in these diagrams represent states in S and the edges represent non-zero transition probabilities. To more completely capture the transitions we can label directed edges with the transition probabilities:



Time evolution

Since P is a matrix, it acts on $\mathbb{R}^{|S|}$ by matrix multiplication. Suppose $\vec{u} \in \mathbb{R}^{|S|}$ with $u_x \geq 0$ for all $x \in S$ and $\sum_x u_x = 1$. Then $\vec{u}$ is a probability distribution on S. For $S = \mathbb{Z}/\ell\mathbb{Z}$, consider $\vec{u}_0 = (1, 0, \ldots, 0)$, indexed as the transition probability matrix is above; this is the probability distribution with $\Pr(S = 0) = 1$ and $\Pr(S \neq 0) = 0$. Then let

$$\vec{u}_1 = P\vec{u}_0 = \begin{pmatrix} 0 & 1-p & & & & p \\ p & 0 & & & & \\ & \ddots & & \ddots & \ddots & \\ & & & & 0 & 1-p \\ 1-p & & & & p & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ \vdots \\ 0 \\ 1-p \end{pmatrix}.$$

This is the probability distribution with $\Pr(S = 1) = p$; $\Pr(S = \ell - 1) = 1 - p$; and $\Pr(S \notin \{1, \ell - 1\}) = 0$, which is the probability distribution for the state after one step of the random walk. Similarly,

$$\vec{u}_2 = P\vec{u}_1 = P \cdot P\vec{u}_0 = P^2\vec{u}_0 = \begin{pmatrix} 2p(1-p) \\ 0 \\ p^2 \\ 0 \\ \vdots \\ 0 \\ (1-p)^2 \\ 0 \end{pmatrix},$$

which is the probability distribution over S after two steps of the random walk. The general result is:

THEOREM. *Let $\vec{u}_0$ be the probability distribution for the initial state X_0 of a Markov process $\{X_t \mid t \in \mathbb{N}\}$. Then the probability distribution for X_t is*

$$\Pr(X_t = x) = (\vec{u}_t)_x = (P^t\vec{u}_0)_x.$$

A conserved quantity

Notice that if ℓ is even, the probability distributions at time 0 and time 2 above have nonzero probabilities only at even numbered states, while at time 1 the probability distribution has nonzero probabilities only at odd numbered states. This is true more generally:

LEMMA. *Let $X_0 = x_0$ for a random walk on $\mathbb{Z}$ or on $\mathbb{Z}/\ell\mathbb{Z}$, ℓ even. Then $x_t + t$ (mod 2) is a "conserved quantity", i.e., it is constant.*

Proof. Consider the change in one timestep: $x_t + t \mapsto (x_t \pm 1) + (t + 1) \in \{x_t + t, x_t + t + 2\}$. But these two values are the same modulo 2, in $\mathbb{Z}$ or $\mathbb{Z}/\ell\mathbb{Z}$, for even ℓ. ∎

This suggests considering a new Markov process with states $S = \{x \in 2\mathbb{Z}\}$, and transition matrix P^2, *i.e.*, two steps of the random walk on $\mathbb{Z}$ we have been considering heretofore:

$$P^2 = \begin{array}{c} \\ \vdots \\ -4 \\ -2 \\ 0 \\ 2 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccc} \dots & -4 & -2 & 0 & 2 & 4 & \dots \end{array} \\ \begin{pmatrix} & & & & & & \\ & \ddots & (1-p)^2 & & & & \\ & & 2p(1-p) & (1-p)^2 & & & \\ & & p^2 & 2p(1-p) & (1-p)^2 & & \\ & & & p^2 & 2p(1-p) & & \\ & & & & p^2 & \ddots & \\ & & & & & & \end{pmatrix} \end{array}.$$

We also will refer to such a Markov process, in which there are nonzero transition probabilities not only to adjacent states, but also to each state itself, as a random walk. Most generally, if there is some geometry on the space of states, and there are only nonzero transitions between nearby states, we will call the Markov process a random walk. In the case $p = 1/2$, the transition probability matrix above becomes

$$B = \begin{array}{c} \\ \vdots \\ -2 \\ 0 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \dots & -2 & 0 & 2 & \dots \end{array} \\ \begin{pmatrix} & & & & \\ & \ddots & 1/4 & & \\ & & 1/2 & & \\ & & 1/4 & \ddots & \\ & & & & \end{pmatrix} \end{array} = \frac{1}{4} \begin{array}{c} \\ \vdots \\ -2 \\ 0 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \dots & -2 & 0 & 2 & \dots \end{array} \\ \begin{pmatrix} & & & & \\ & \ddots & 1 & & \\ & & 2 & & \\ & & 1 & \ddots & \\ & & & & \end{pmatrix} \end{array},$$

which we call B for "binomial". Setting $\vec{u}_0 \in \mathbb{R}^{2\mathbb{Z}}$ to be 1 only in the 0^{th} component, the probability distribution after t steps of this new random walk is

$$\vec{u}_t = B^t \vec{u}_0.$$

We will refer to the original random walk as $\mathrm{RW}^1(p)$, and to the random walk with transition probability matrix P^2 as $\mathrm{RW}^2(p)$.