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Random Walk Algorithms: Lecture 7

David A. Meyer

In which the heat equation with drift in one dimension is derived as a continuum limit of a random walk on $2\mathbb{Z}$, and the advection equation is solved.

Difference equations

Having constructed a new random walk, $\mathrm{RW}^2(p)$, from two steps of the original random walk, $\mathrm{RW}^1(p)$, in the last lecture, we have an equation for the evolution of the probability distribution over $2\mathbb{Z}$ during one timestep:

$$\vec{u}_{t+1} = B\vec{u}_t.$$

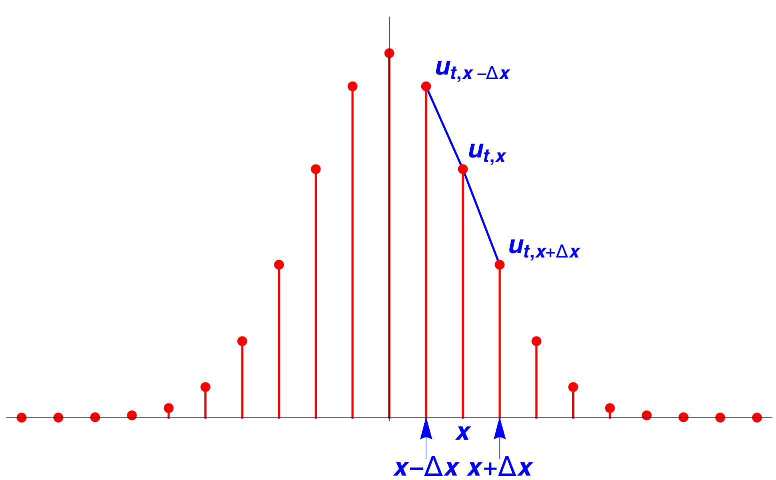
Written as an equation for the change in the probability distribution, this becomes:

$$\vec{u}_{t+1} - \vec{u}_t = (B-I)\vec{u}_t = \frac{1}{4} \begin{array}{cc} & \begin{matrix} \ldots & -2 & 0 & 2 & \ldots \end{matrix} \\ \begin{matrix} \vdots \\ -2 \\ 0 \\ 2 \\ \vdots \end{matrix} & \begin{pmatrix} & & & & \\ & \ddots & 1 & & \\ & & -2 & & \\ & & 1 & \ddots & \\ & & & & \end{pmatrix} \end{array} \vec{u}_t = \frac{1}{4} L\vec{u}_t,$$

where L is called the *graph Laplacian* and is defined as the *adjacency matrix* for a graph (the matrix A indexed by the vertices of a graph with $A_{ij} = 1$ if there is an edge connected i and j and $A_{ij} = 0$ otherwise) minus the diagonal matrix D with D_{ii} being the degree of i: $L = A - D$. Written in terms of components, this becomes a set of coupled difference equations:

$$u_{t+1,x} - u_{t,x} = \frac{1}{4}(u_{t,x+2} - 2u_{t,x} + u_{t,x-2}).$$

For each t, $\{u_{t,x} \mid x \in 2\mathbb{Z}\}$ is a probability distribution. We can imagine that it is a discrete set of values of a differentiable function. The two blue line segments shown not only approximate the function on the intervals $[x-\Delta x, x]$ and $[x, x+\Delta x]$, but their slopes also approximate the left and right derivatives at x.



Derivatives

Recall that the right and left derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ are defined as the limits:

$$
\begin{aligned}
f'_+(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}; \\
f'_-(x) &= \lim_{\Delta x \to 0} \frac{f(x) - f(x-\Delta x)}{\Delta x}.
\end{aligned}
$$

If these two limits are equal, then the function is differentiable at x. Supposing f is also second differentiable at x, its second derivative is

$$
\begin{aligned}
f''(x) &= \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\big(f(x+\Delta x) - f(x)\big)/\Delta x - \big(f(x) - f(x-\Delta x)\big)/\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}
\end{aligned}
$$

If the duration of each timestep is Δt, and the length of each spatial unit is Δx, we can rewrite the difference equation above as:

$$
u_{t+\Delta t,x} - u_{t,x} = \frac{1}{4}(u_{t,x+\Delta x} - 2u_{t,x} + u_{t,x-\Delta x}),
$$

whence the definitions of derivatives suggest setting $\Delta t = (\Delta x)^2/(4\alpha)$, $0 < \alpha \in \mathbb{R}$, dividing, and taking the limit as $\Delta t \to 0$:

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{u_{t+\Delta t,x} - u_{t,x}}{\Delta t} &= \lim_{\Delta x \to 0} \frac{1}{4} \frac{u_{t,x+\Delta x} - 2u_{t,x} + u_{t,x-\Delta x}}{(\Delta x)^2/(4\alpha)} \\
\Rightarrow \quad \frac{\partial u(t,x)}{\partial t} &= \alpha \frac{\partial^2 u(t,x)}{\partial x^2}.
\end{aligned}
$$

This is the *diffusion* (or *heat*) equation.

Drift

It is only for $p = 1/2$ that two steps of $\mathrm{RW}^1(p)$ are described by the transition matrix B; otherwise, let $p = 1/2 + \epsilon$, for $-1/2 \le \epsilon \le 1/2$. Then the elements in the two step transition probability matrix P^2 are:

$$
\begin{aligned}
p^2 &= (\frac{1}{2} + \epsilon)^2 = \frac{1}{4} + \epsilon^2 + 2\epsilon \\
2p(1-p) &= 2(\frac{1}{2} + \epsilon)(\frac{1}{2} - \epsilon) = \frac{1}{2} - 2\epsilon^2 \\
(1-p)^2 &= (\frac{1}{2} - \epsilon)^2 = \frac{1}{4} + \epsilon^2 - 2\epsilon,
\end{aligned}
$$

so the difference equations become

$$u_{t+1,x} - u_{t,x} = (\frac{1}{4} + \epsilon^2)(u_{t,x+2} - 2u_{t,x} + u_{t,x-2}) - 2\epsilon(u_{t,x+2} - u_{t,x-2}).$$

As before we want to divide the left hand side by Δt, and the first term on the right hand side by $(\frac{1}{4} + \epsilon^2)(\Delta x)^2/\alpha$. For the last term to become a derivative in the limit we must divide it by $2\Delta x$ (since it contains $u_{t,x+\Delta x} - u_{t,x-\Delta x}$, and to clean up the coefficient in front of the derivative we will divide by $2\epsilon \cdot 2\Delta x/\beta$. Thus we must set

$$\Delta t = (\frac{1}{4} + \epsilon^2)\frac{1}{\alpha}(\Delta x)^2 = \frac{4\epsilon}{\beta}\Delta x,$$

(which means β has the same sign as ϵ) divide by this quantity, and take the limit as it goes to 0. (Notice that this means that $\epsilon \to 0$.) The result is the heat equation with *drift*:

$$\frac{\partial u(t,x)}{\partial t} = \alpha\frac{\partial^2 u(t,x)}{\partial x^2} - \beta\frac{\partial u(t,x)}{\partial x}.$$

The advection equation

Setting $\alpha = 0$ in the heat equation with drift gives the *advection* equation:

$$\frac{\partial u(t,x)}{\partial t} = -\beta\frac{\partial u(t,x)}{\partial x}.$$

If we define new variables $(\tau, \chi) = (t, x - \beta t)$, then

$$\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \tau} = \frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} = 0,$$

using the chain rule, and then the advection equation. Thus

$$u(t,x) = u\big(t(0,\chi), x(0,\chi)\big) = u(0,\chi) = u(0, x - \beta t),$$

so whatever the initial function $u(0,x)$ is, at time t it is merely shifted by βt.