168A MIDTERM

In problem 1, write pseudocode for your algorithms; you can also describe in words what they are doing. In problems 2 and 3, showing your work in addition to the final answer might earn some partial credit in case the latter is not exactly right.

1. a. [15 points] Construct an algorithm to select a point uniformly at random from the disk of radius 1 centered at the origin. Assume you have access to a random number generator random() that returns a uniformly random number in the interval $[0,1]$, *e.g.*, $x \leftarrow$ random() means that $x \in [0,1]$, sampled from the constant probability density function.

   b. [20 points] Now construct an algorithm to approximate the area of a disk of radius 1, *i.e.*, to approximate π. Design your algorithm so that it has probability at least $2/3$ of giving an estimate for π that is correct to one decimal place, *i.e.*, it is within the interval $[\pi - 0.05, \pi + 0.05]$.
   Hint: For large n and fixed p, you can approximate a binomial distribution by a normal distribution with the same mean and variance, *i.e.*, $\text{Binomial}(n,p) \approx \mathcal{N}\big(np, np(1-p)\big)$.

2. Consider a random walk $\{X_t \in \mathbb{Z} \mid t \in \mathbb{N}\}$ with transition probabilities

$$P_{xy} = \begin{cases} 1/2 & \text{if } |x-y| = 1; \\ 0 & \text{otherwise,} \end{cases}$$

   and initial state $X_0 = 0$.

   a. [15 points] Suppose $X_4 = 2$. Draw all the paths in (x,t) space that the walk could follow.
   Hint: Each path is like the path that you plotted in the first homework assignment.

   b. [20 points] Compute the probability distributions for X_1, X_2, and X_3 given that $X_0 = 0$ and $X_4 = 2$, *i.e.*, compute $\Pr(X_t = x \mid X_0 = 0, X_4 = 2)$ for $t \in \{1,2,3\}$.

3. a. [15 points] Write the transition probability matrix P for a random walk on $S = \{1,2,3,4,5,6\} \subset \mathbb{N}$ with transition probabilities:

$$\Pr(X_{t+1} = x \mid X_t = y) = \begin{cases} 1/2 & \text{if } x = y; \\ 1/2 & \text{if } y \in \{1,6\} \text{ and } |x-y| = 1; \\ 1/4 & \text{if } y \notin \{1,6\} \text{ and } |x-y| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

   b. [15 points] Find a probability distribution on S that is unchanged by the evolution.