UC San Diego

Pentagramma mirificum

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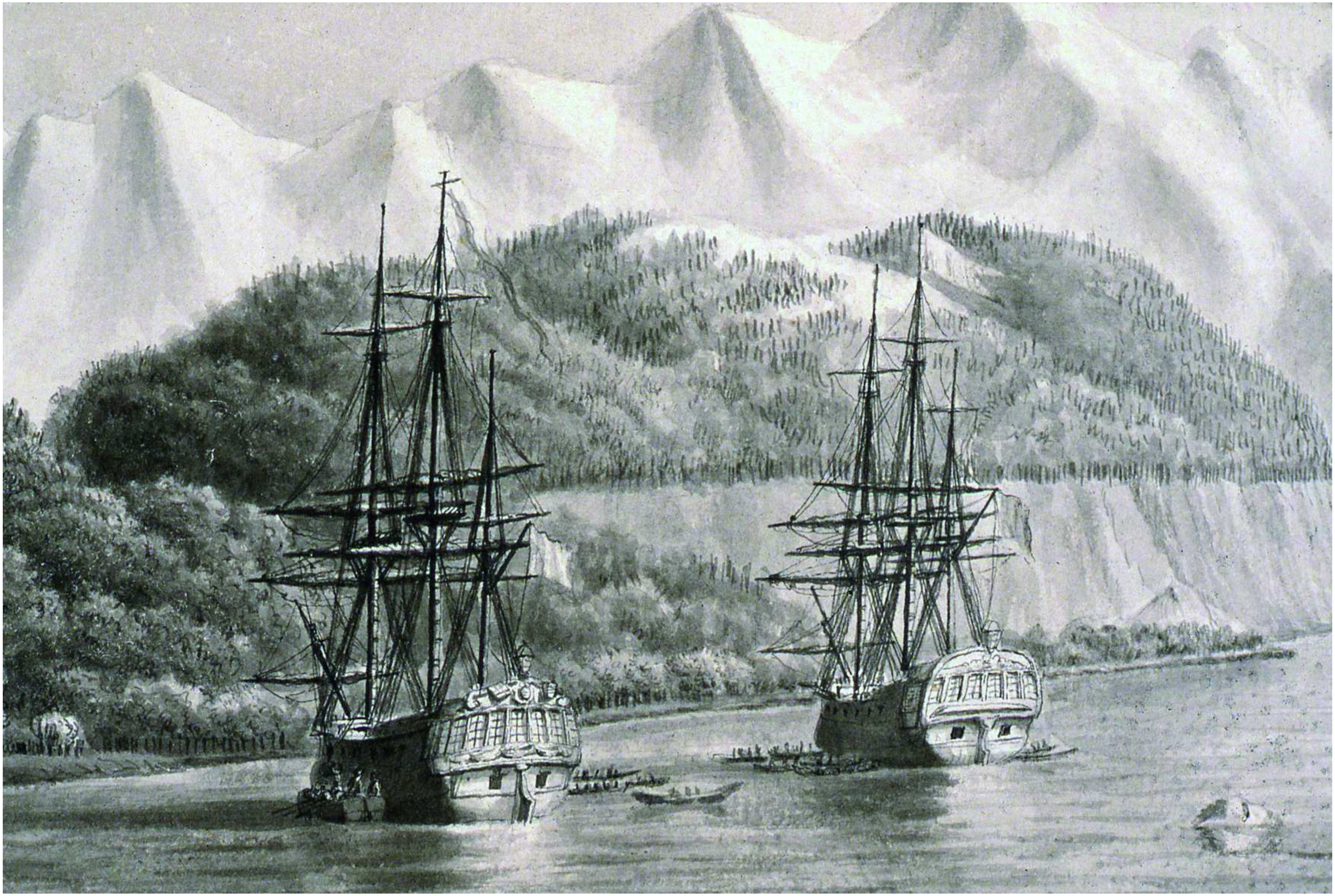
San Diego Math Circle
Gauss and Cauchy groups
UC San Diego, La Jolla, CA
28 September 2019



Frigates l'*Astrolabe* and la *Boussole* during the expedition of La Pérouse (1786)

George Hamilton Brodhead, Mission San Carlos Borromeo de Carmelo (1890)


humilitas

supérieurs. J'avoue que, plus ami des droits de l'homme que théologien, j'aurais désiré qu'aux principes du christianisme, on eût joint une législation qui, peu-à-peu, eût rendu citoyens, des hommes dont l'état ne diffère presque pas aujourd'hui de celui des nègres des habitations de nos colonies, régies avec le plus de douceur et d'humanité.

1786.
SEPTEMBRE.

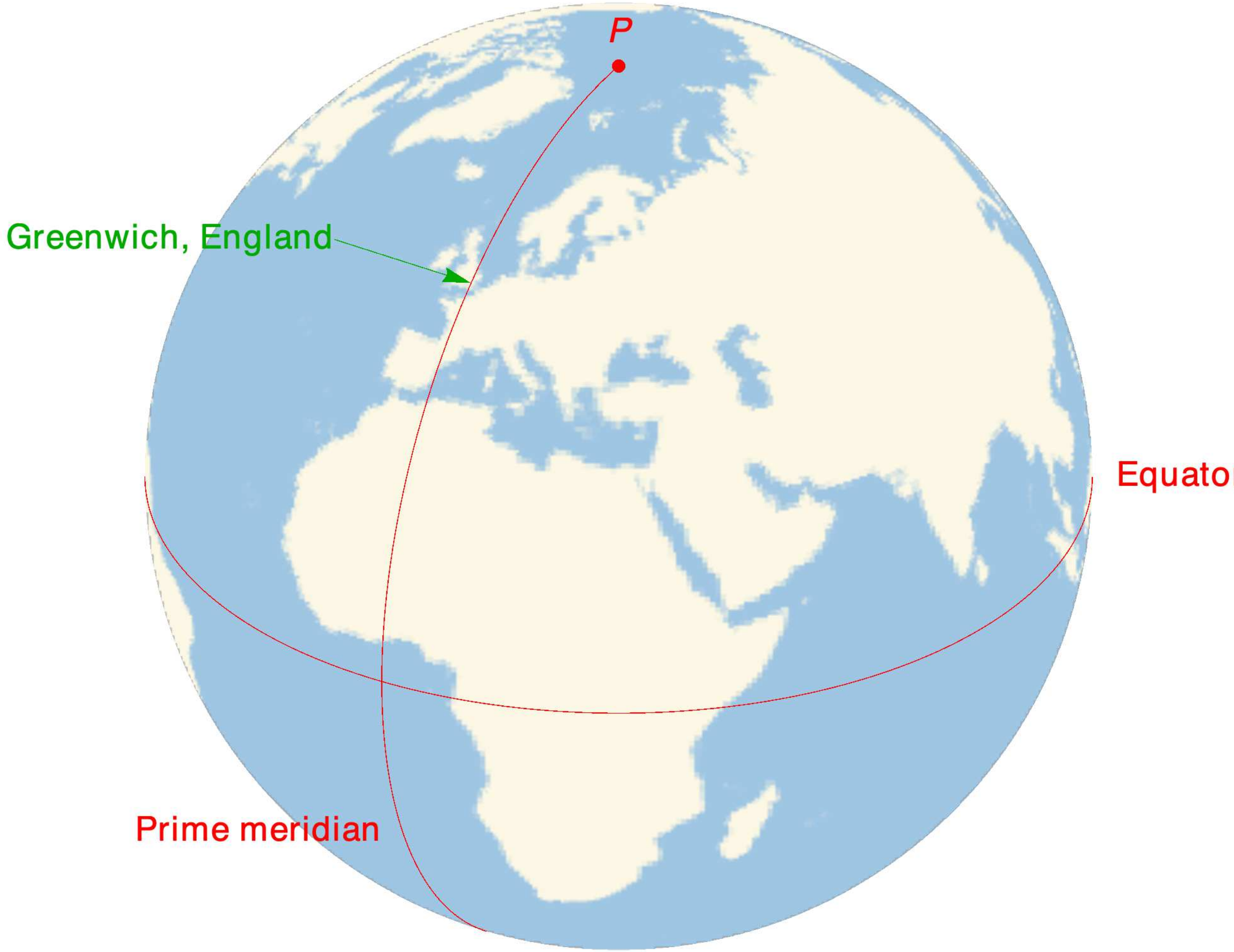
Les voyages de divers navigateurs anglais, en étendant nos connaissances, avaient mérité la juste admiration du monde entier : l'Europe avait apprécié les talens et le grand caractère du capitaine COOK. Mais dans un champ aussi vaste, il restera pendant bien des siècles de nouvelles connaissances à acquérir; des côtes à relever; des plantes, des arbres, des poissons, des oiseaux à décrire; des minéraux, des volcans à observer; des peuples à étudier, et peut-être à rendre plus heureux : car enfin, une plante farineuse, un fruit de plus, sont des bienfaits inestimables pour les habitans des îles de la mer du Sud [c].

1785. AOÛT.

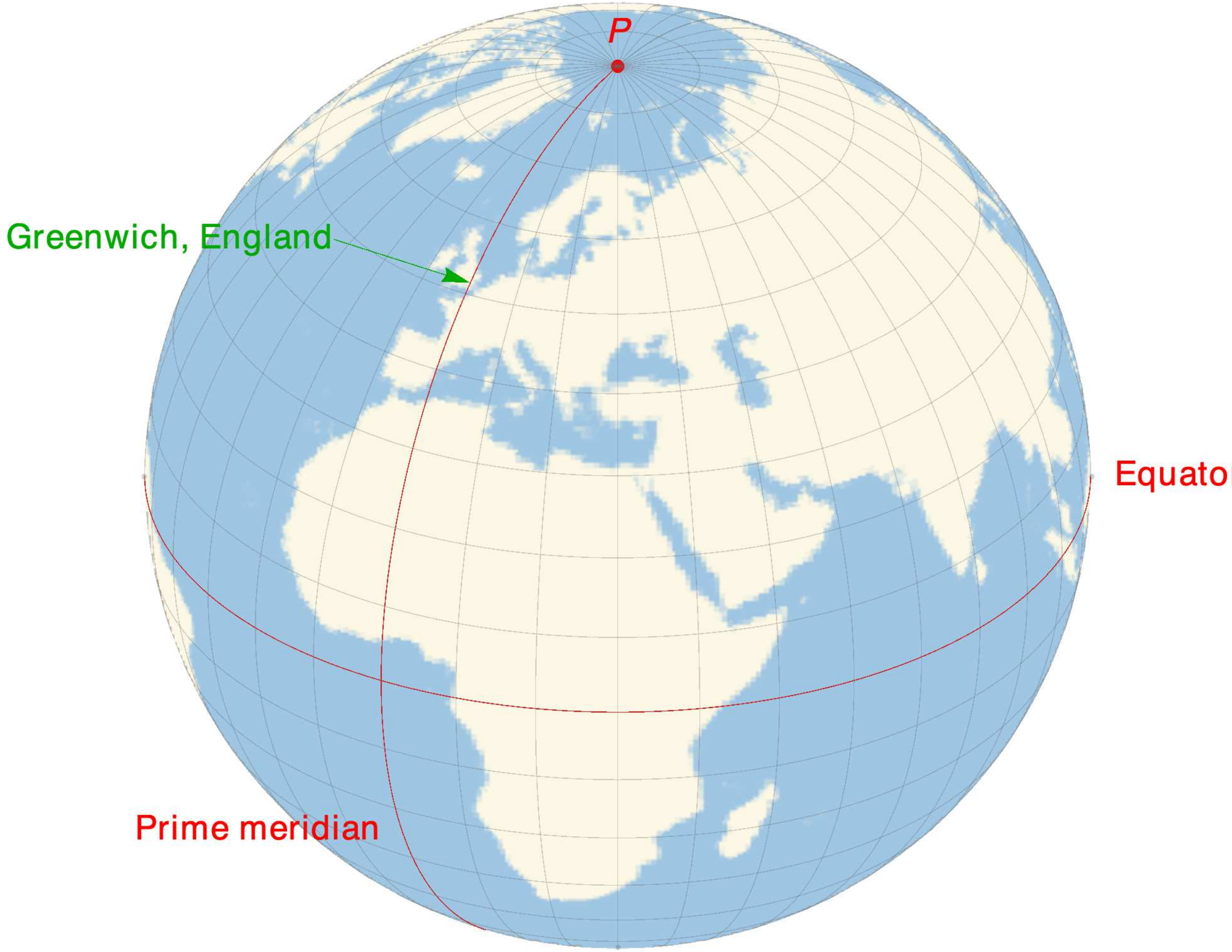
Ces différentes réflexions firent adopter le projet d'un voyage autour du monde; des savans de tous les genres furent employés dans cette expédition. M. DAGELET, de l'académie des sciences, et M. MONGE [d], l'un et l'autre professeurs de mathématiques à l'École militaire, furent embarqués en qualité d'astronomes; le premier sur la BOUSSOLE, et le second sur l'ASTROLABE. M. DE LAMANON, de l'académie

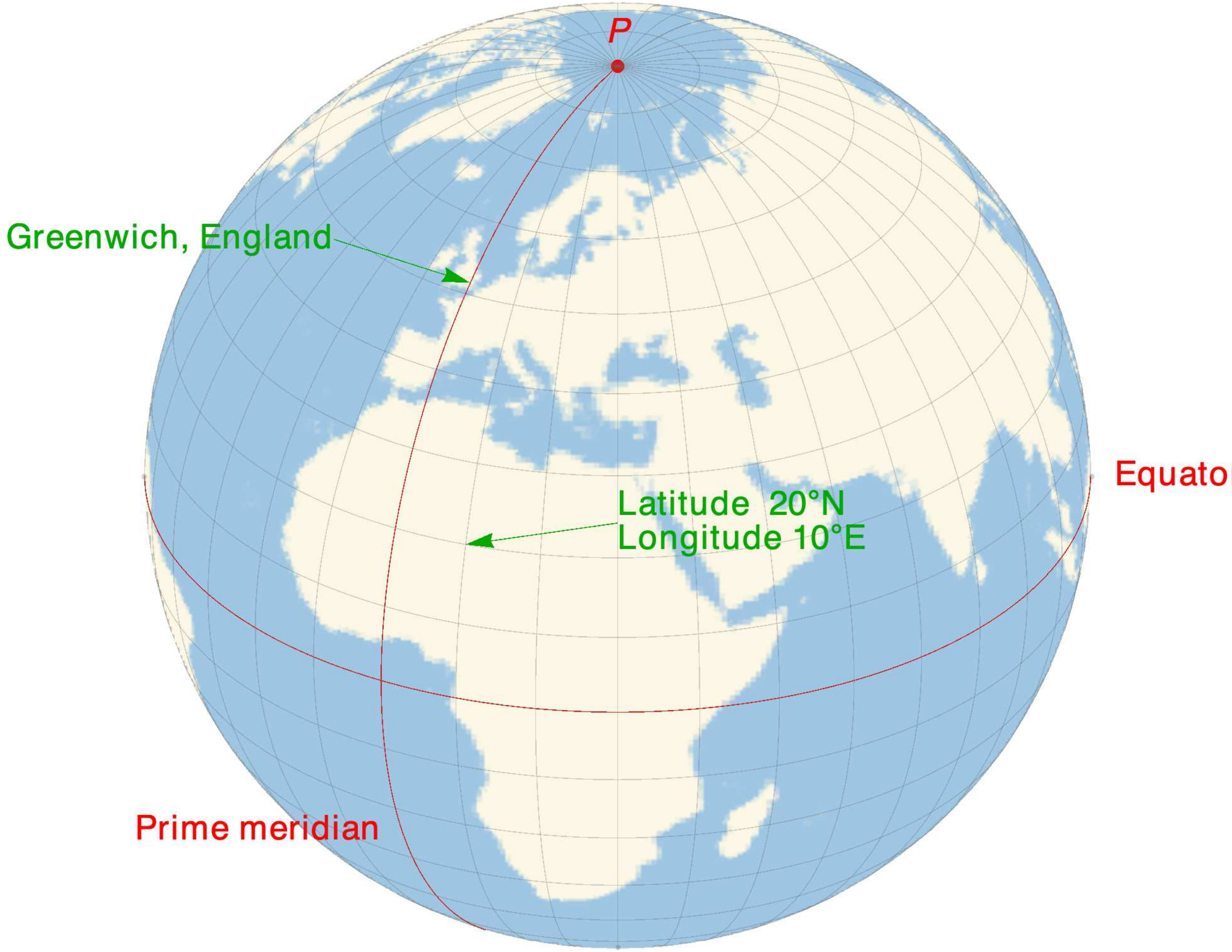

P


P
Equator



P
Greenwich, England
Equator
Prime meridian


P
Greenwich, England
Equator
Prime meridian


P
Greenwich, England
Equator
Latitude 20°N
Longitude 10°E
Prime meridian

1785. AOÛT.

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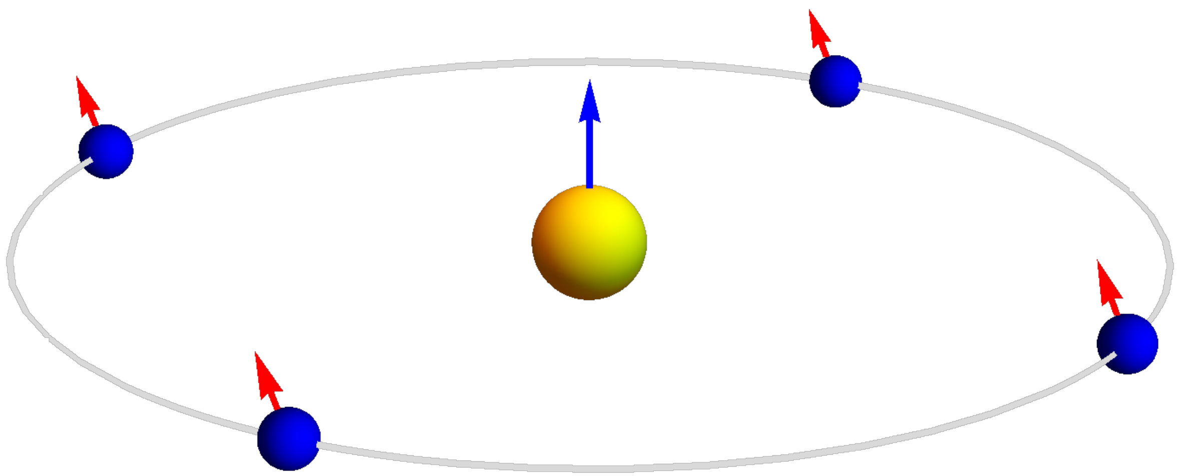

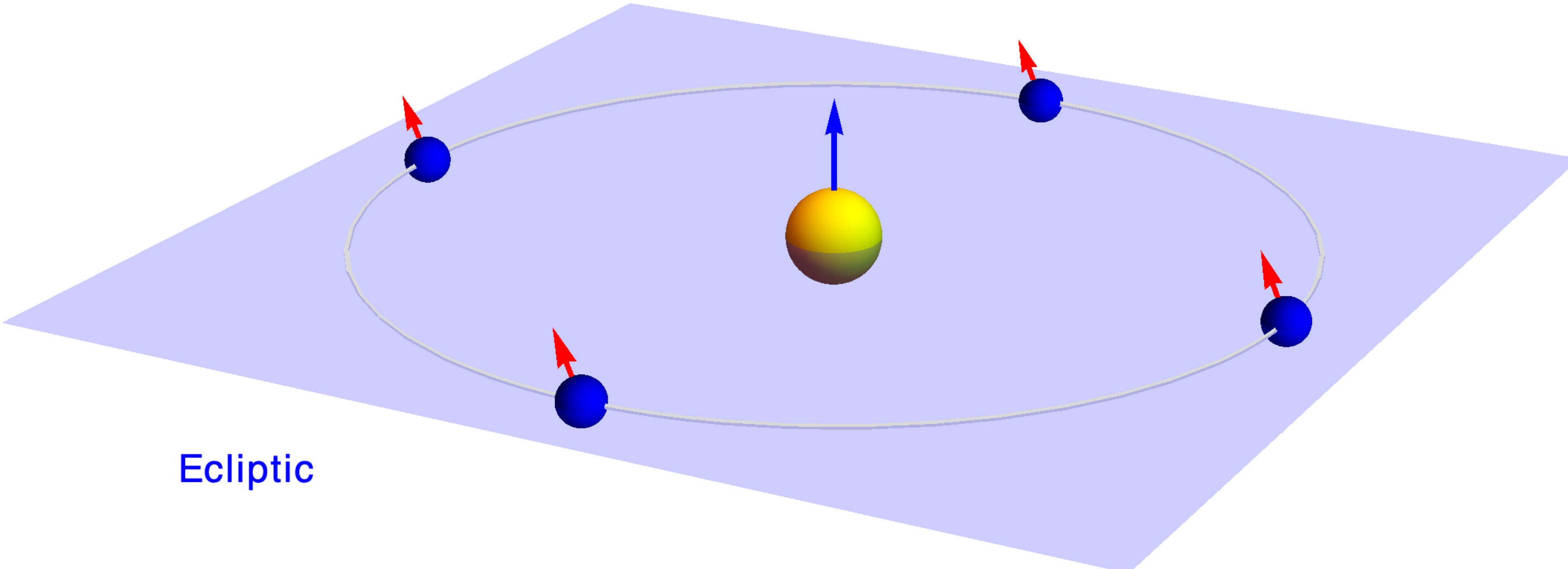
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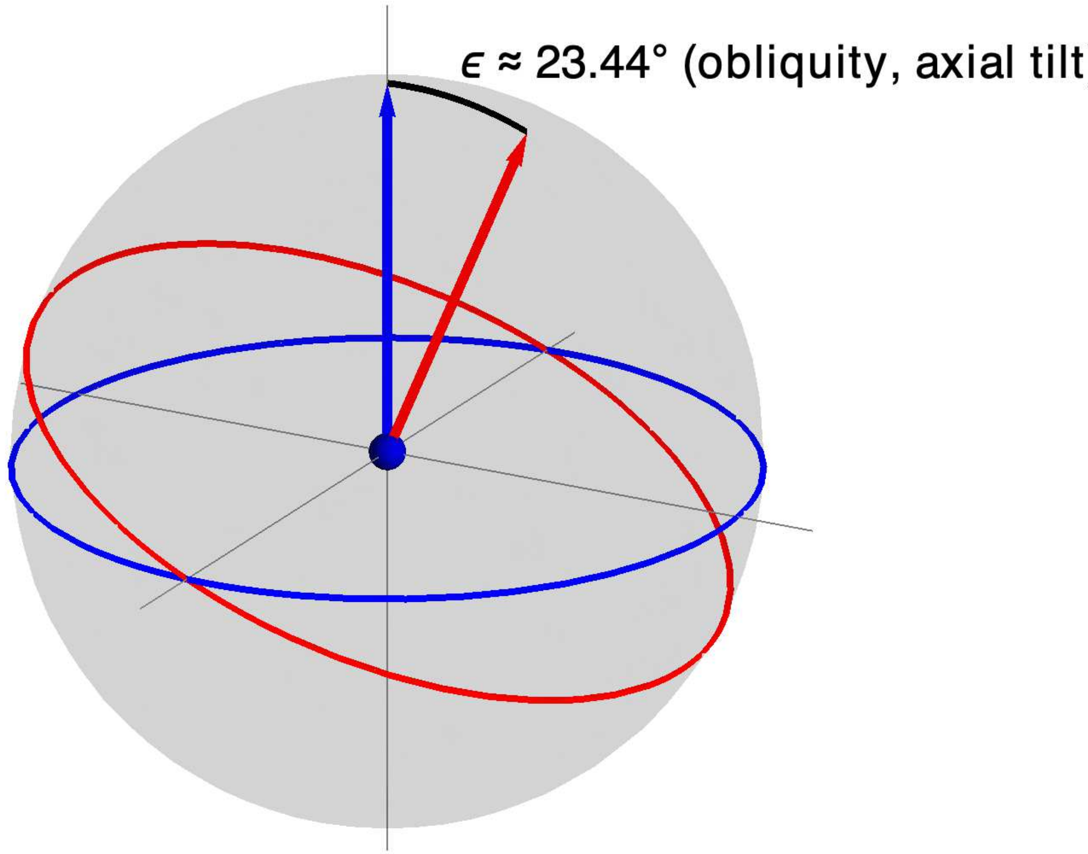
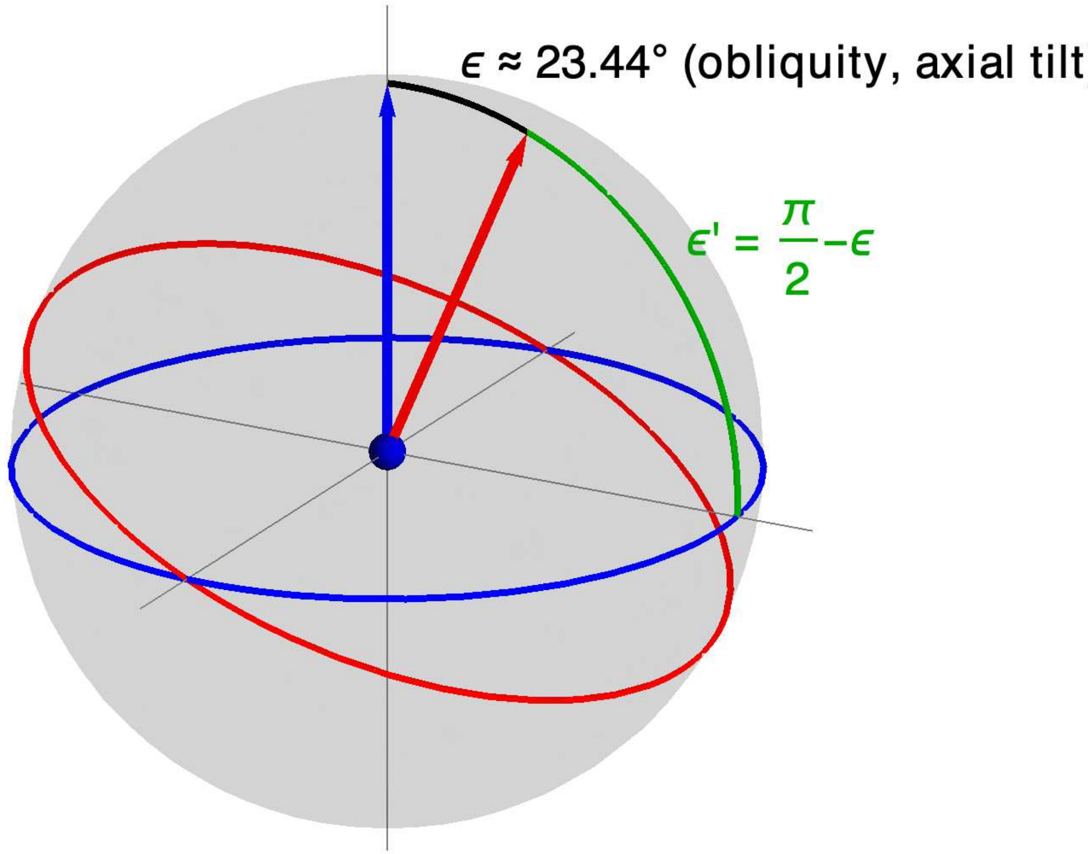
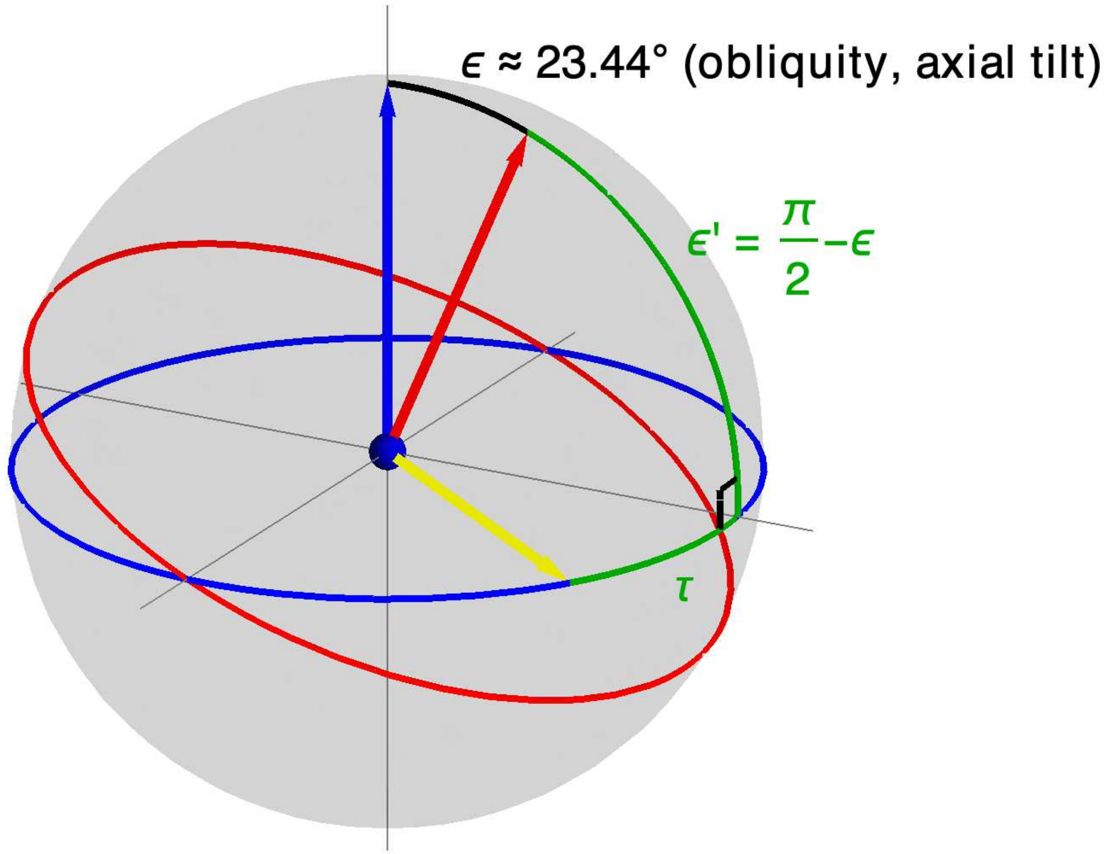
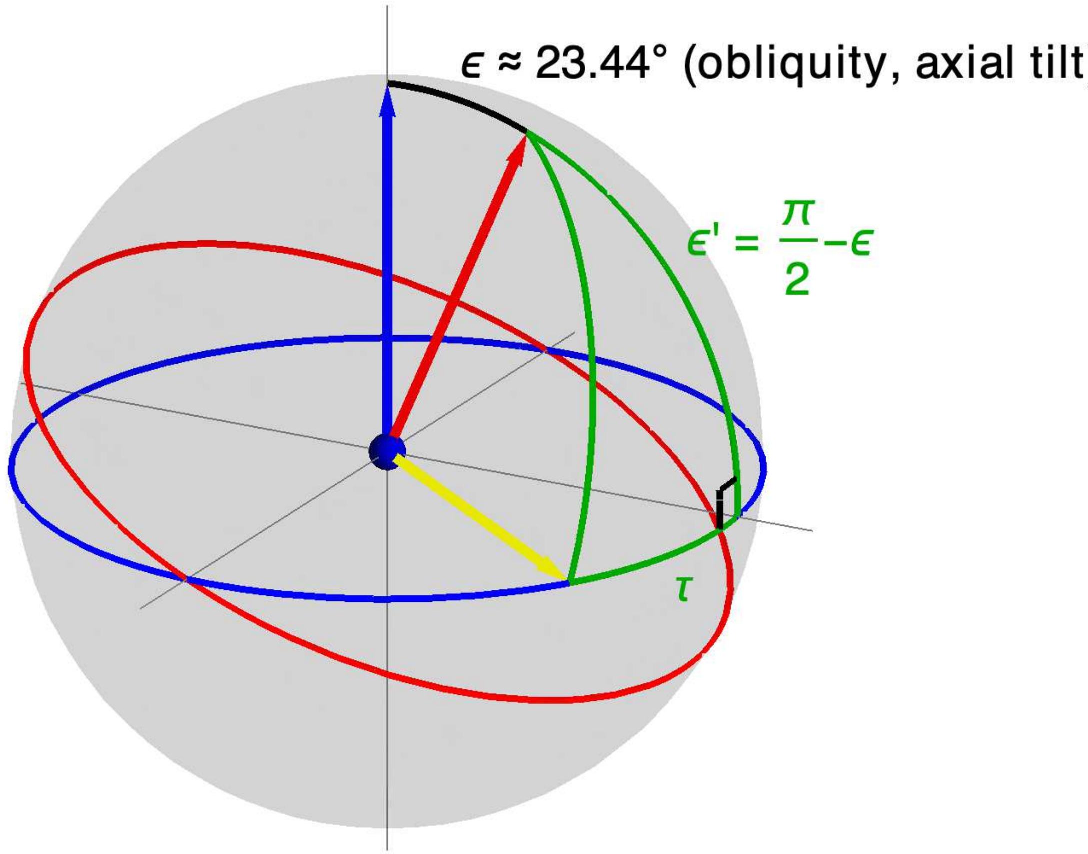


$\epsilon \approx 23.44°$ (obliquity, axial tilt)

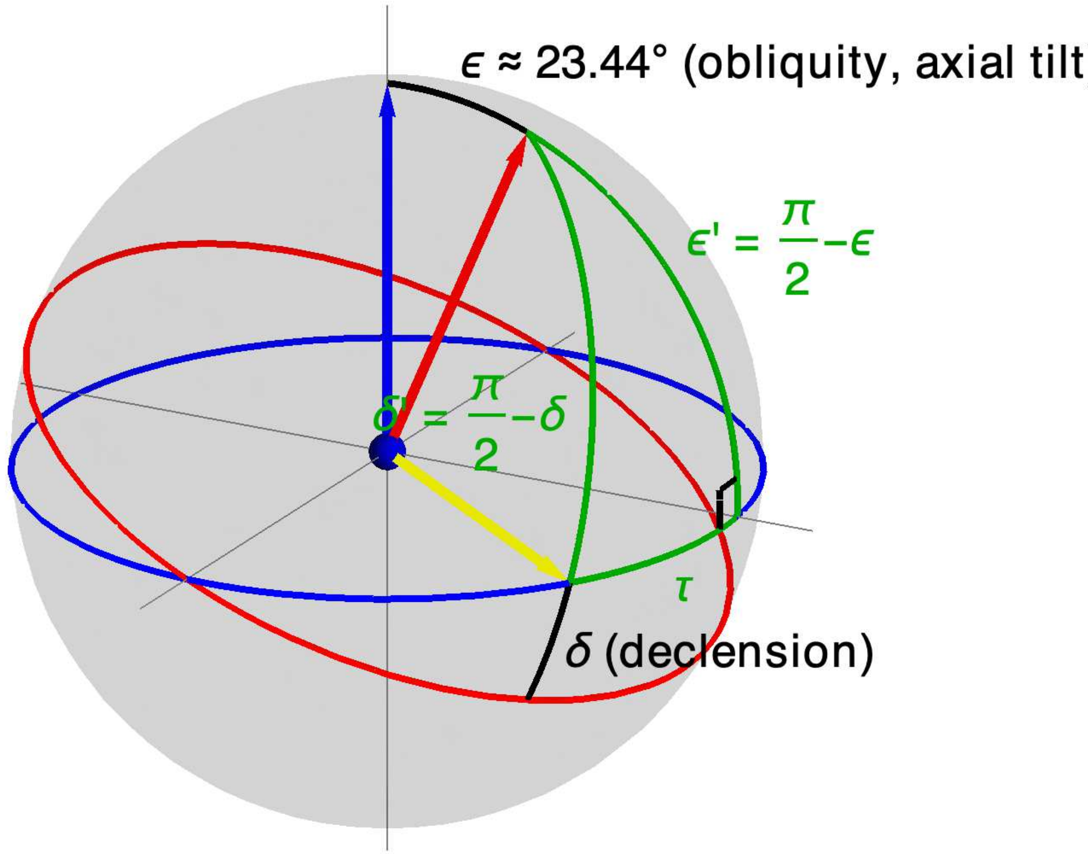


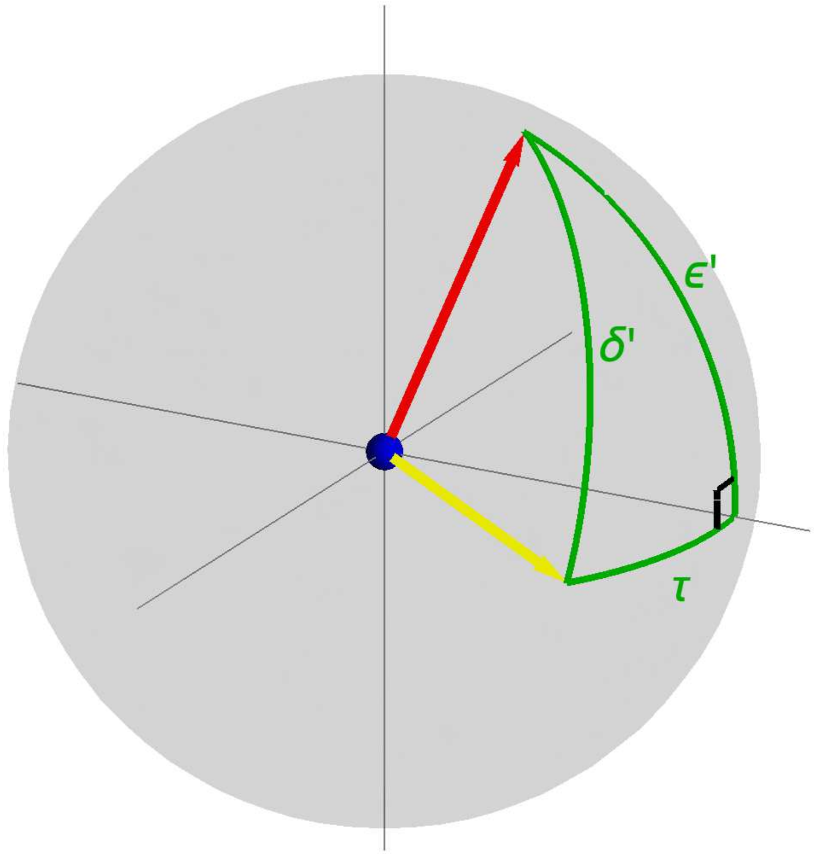
$\epsilon \approx 23.44°$ (obliquity, axial tilt)
$\epsilon' = \frac{\pi}{2} - \epsilon$


ϵ ≈ 23.44° (obliquity, axial tilt)
ϵ' = π/2 − ϵ
τ



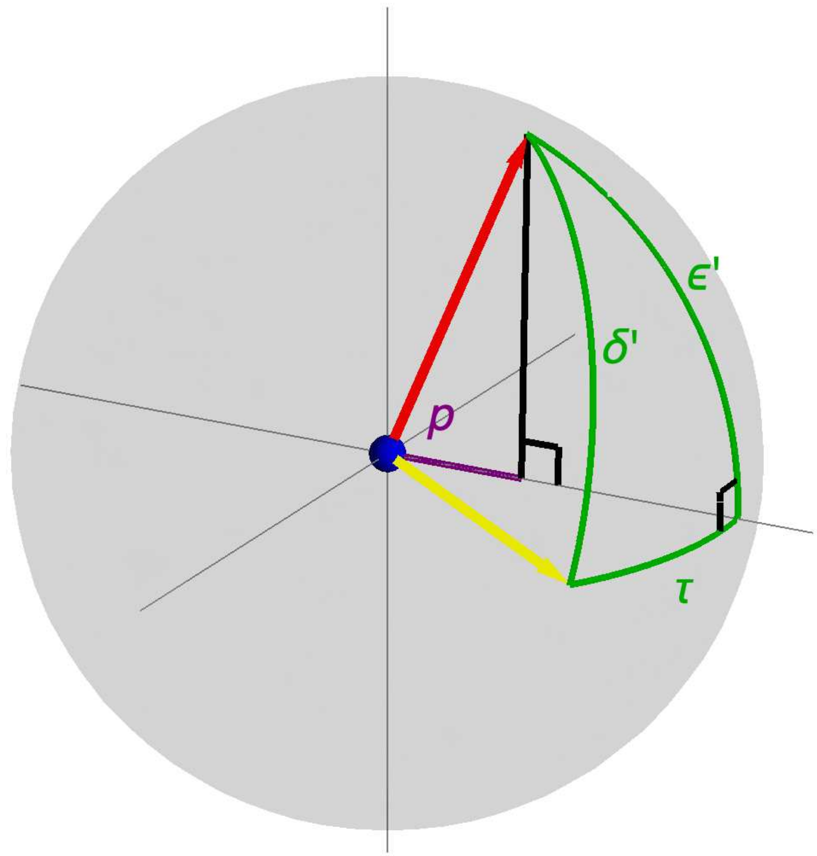
ϵ ≈ 23.44° (obliquity, axial tilt)
ϵ' = π/2 − ϵ
τ


$\epsilon \approx 23.44°$ (obliquity, axial tilt)
$\epsilon' = \frac{\pi}{2} - \epsilon$
$\delta' = \frac{\pi}{2} - \delta$
τ
δ (declension)

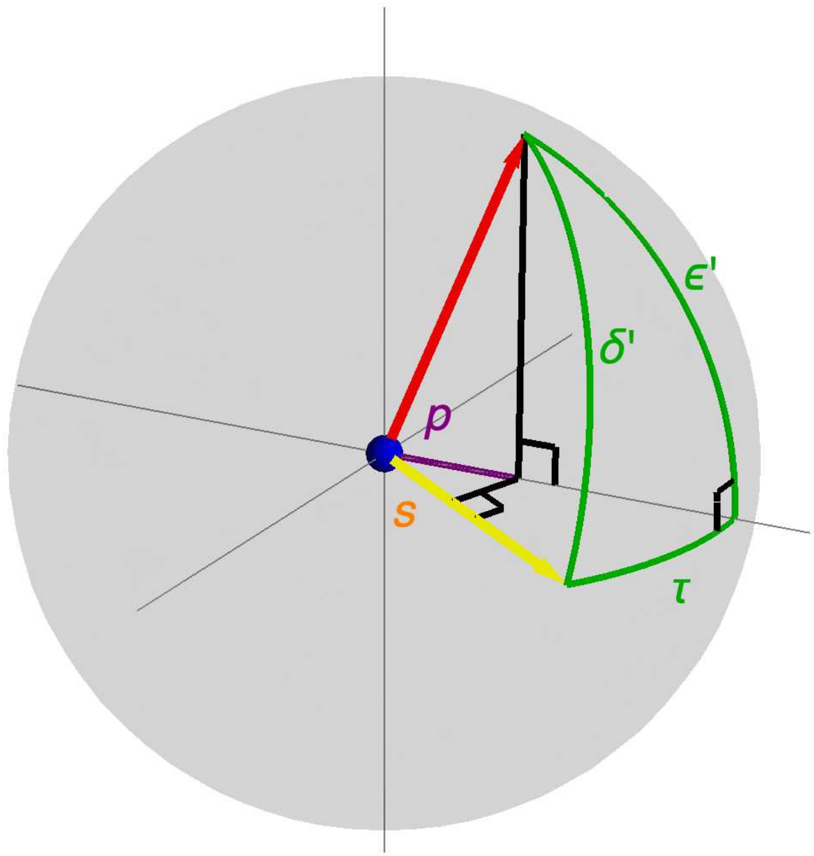


(right) spherical triangle

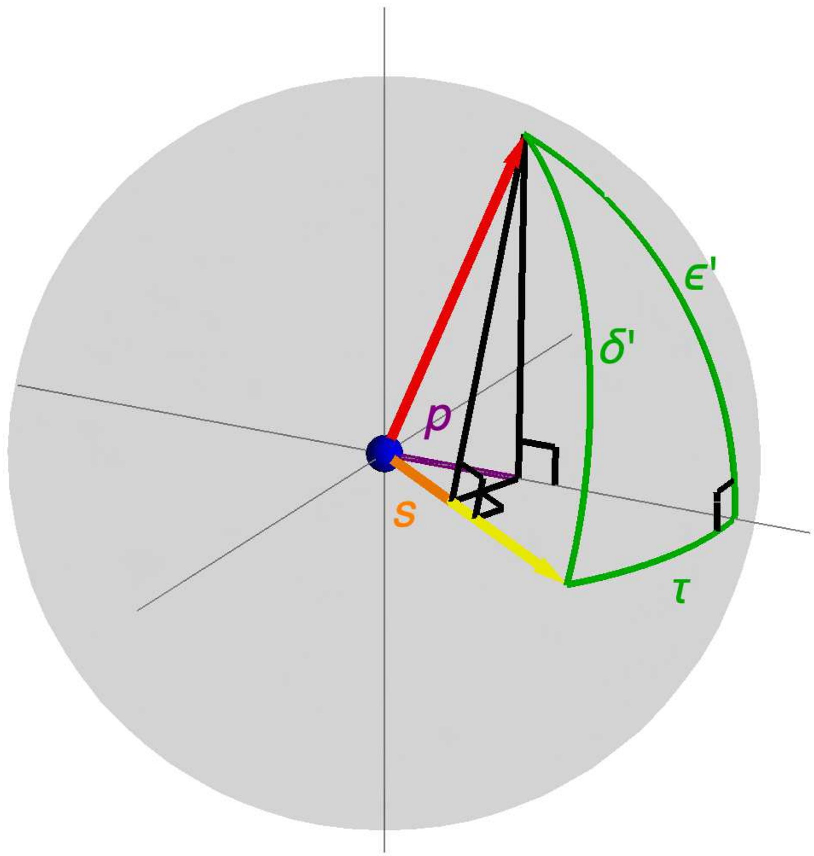
Sides are arcs of great circles.



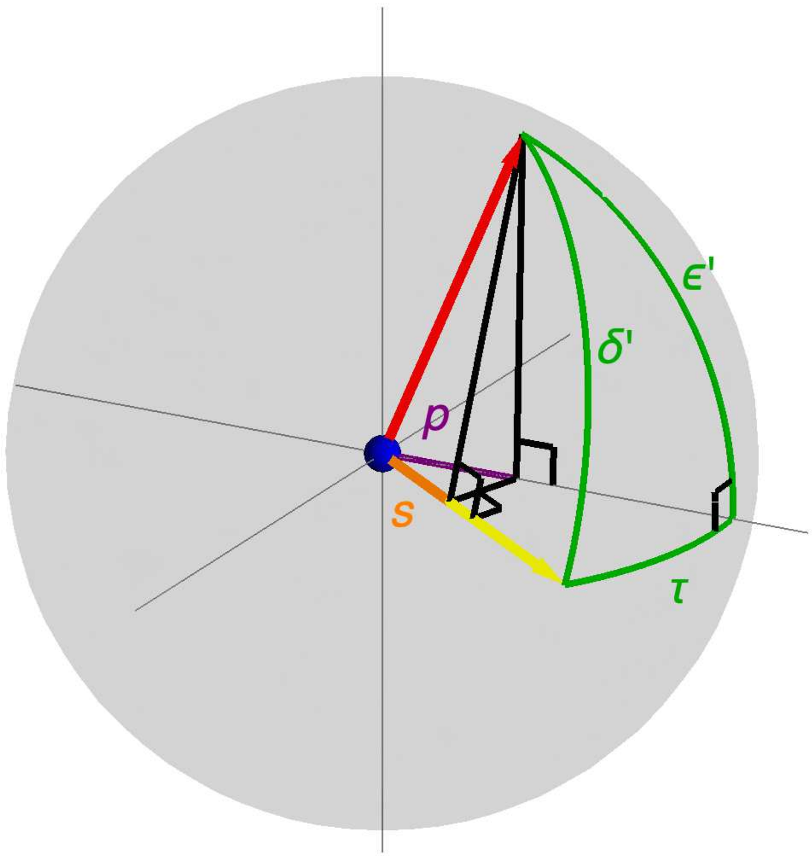
$$p = \cos \epsilon'$$



$p = \cos\epsilon'$; $s = p\cos\tau$



$$p = \cos\epsilon';\ s = p\cos\tau;\ s = \cos\delta'$$



$$p = \cos\epsilon';\ s = p\cos\tau;\ s = \cos\delta'$$

$$\Rightarrow \cos\delta' = \cos\epsilon'\cos\tau$$

Right spherical triangles

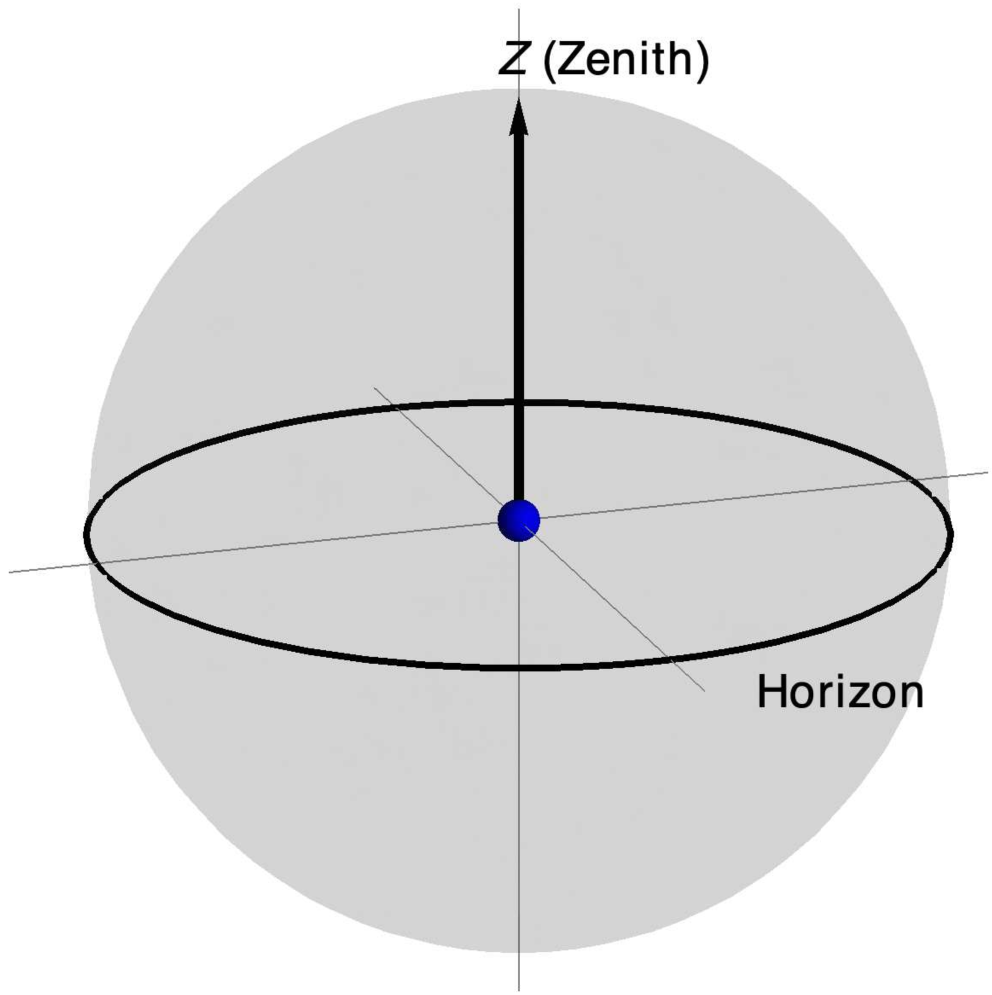
Let $\triangle ABC$ be a spherical triangle with sides α, β and γ. If C is a right angle, then $\triangle ABC$ is a right spherical triangle.

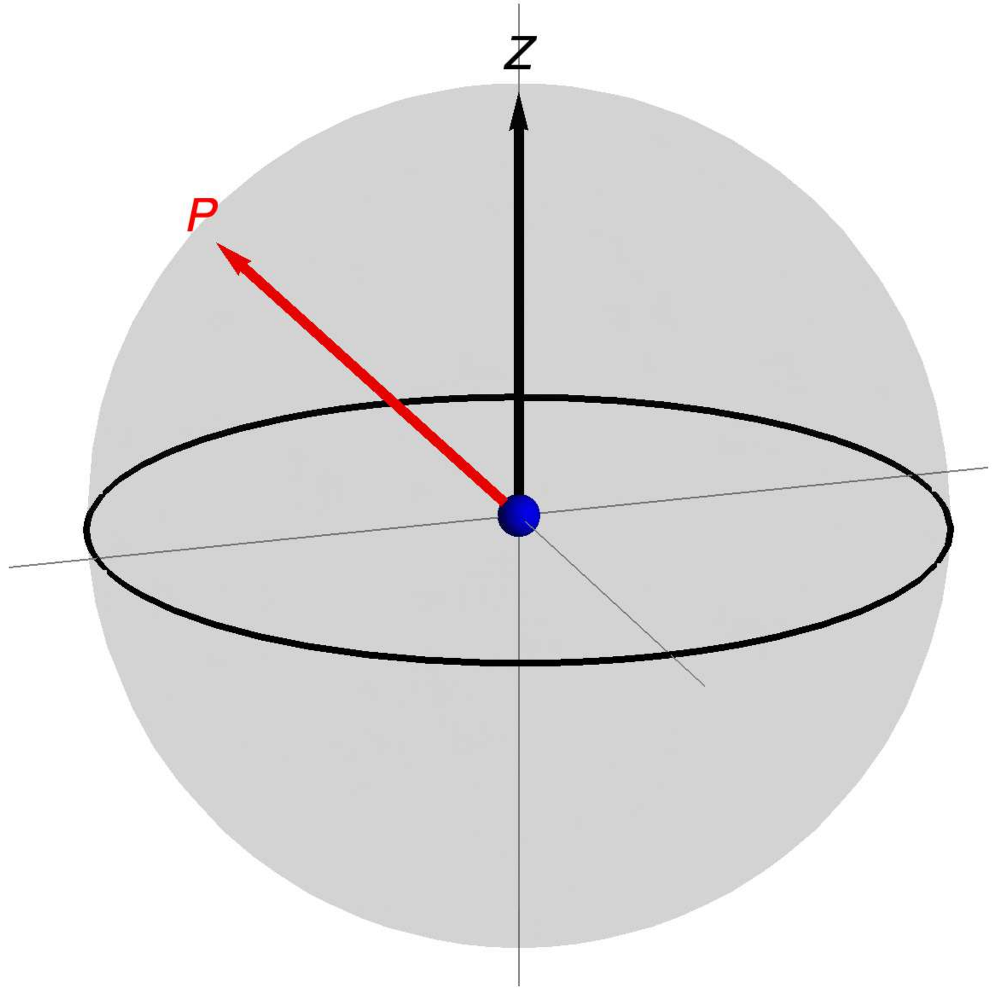
We just showed that

$$\cos\gamma = \cos\alpha\cos\beta.$$

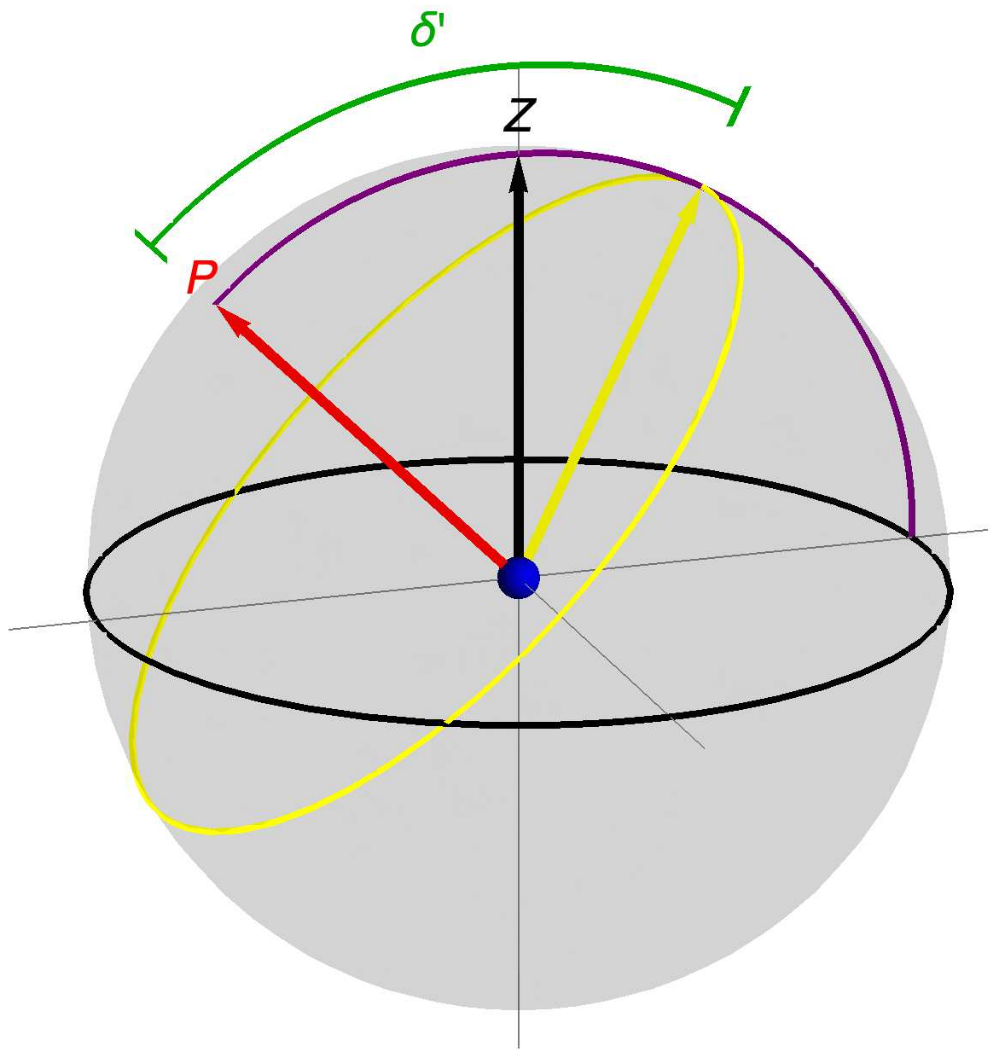
A similar argument [exercise] shows that

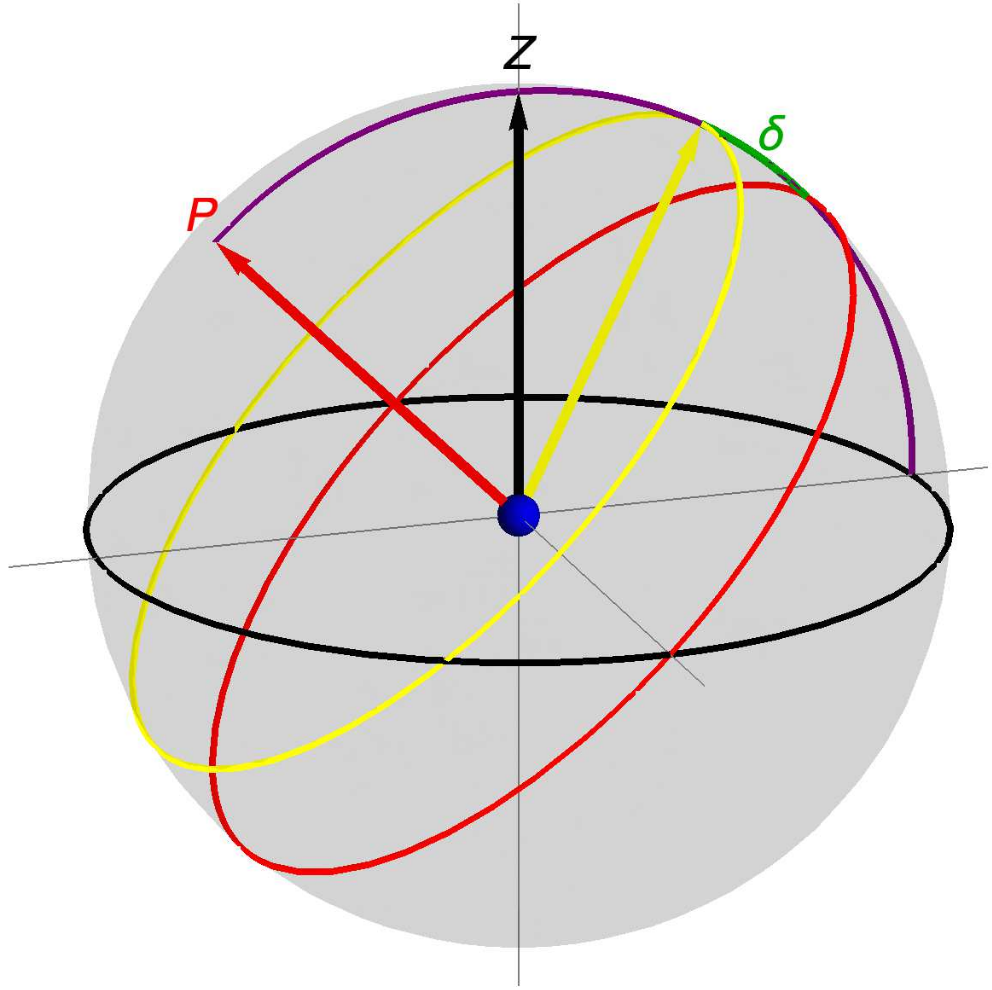
$$\cos A = \tan\beta\cot\gamma.$$

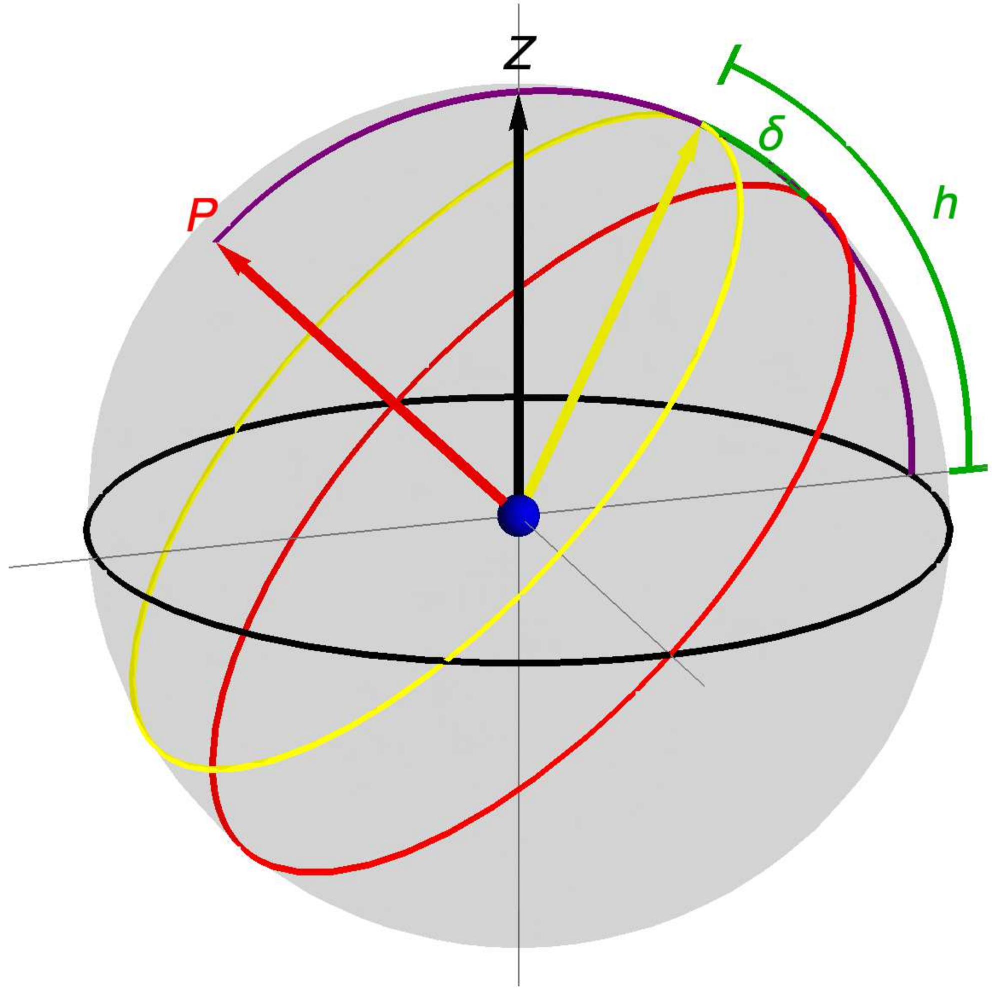

Z (Zenith)
Horizon

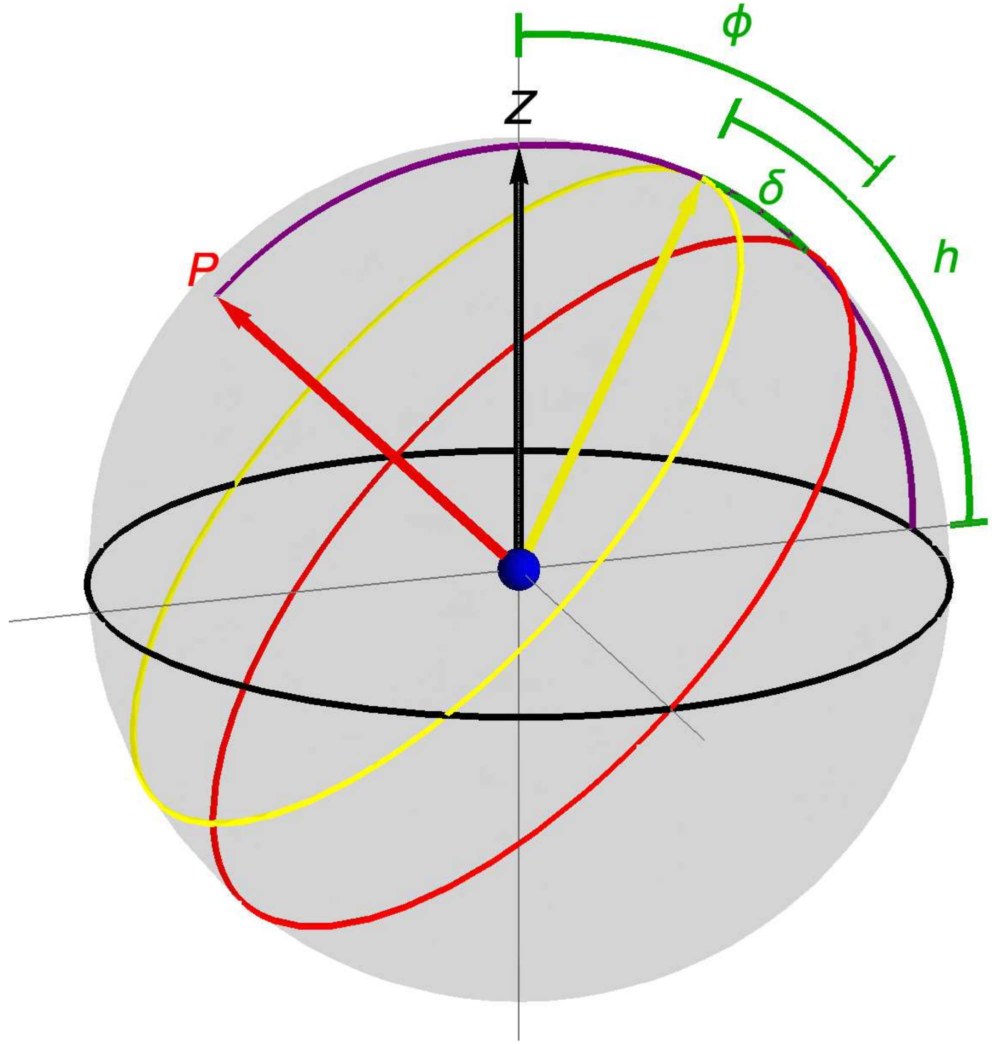


Z
P


δ'
Z
P


Z
P
δ


Z
P
δ
h



latitude $\phi = \frac{\pi}{2} - (h - \delta)$

celle obtenue par des distances. C'est d'après ces opérations que nous avons déterminé la position en longitude des îles Martin-Vas et de l'île de la Trinité. Nous avons aussi déterminé très-soigneusement les latitudes, non seulement en observant avec exactitude la hauteur méridienne du soleil,

1785. Octobre. mais en prenant un très-grand nombre de hauteurs près du méridien, et en les réduisant toutes à l'instant du midi vrai, conclu par des hauteurs correspondantes. Les erreurs les plus fortes que nous ayons pu avoir par cette méthode, n'excèdent pas 20″.

Anno Duodecimo

Annæ Reginæ.

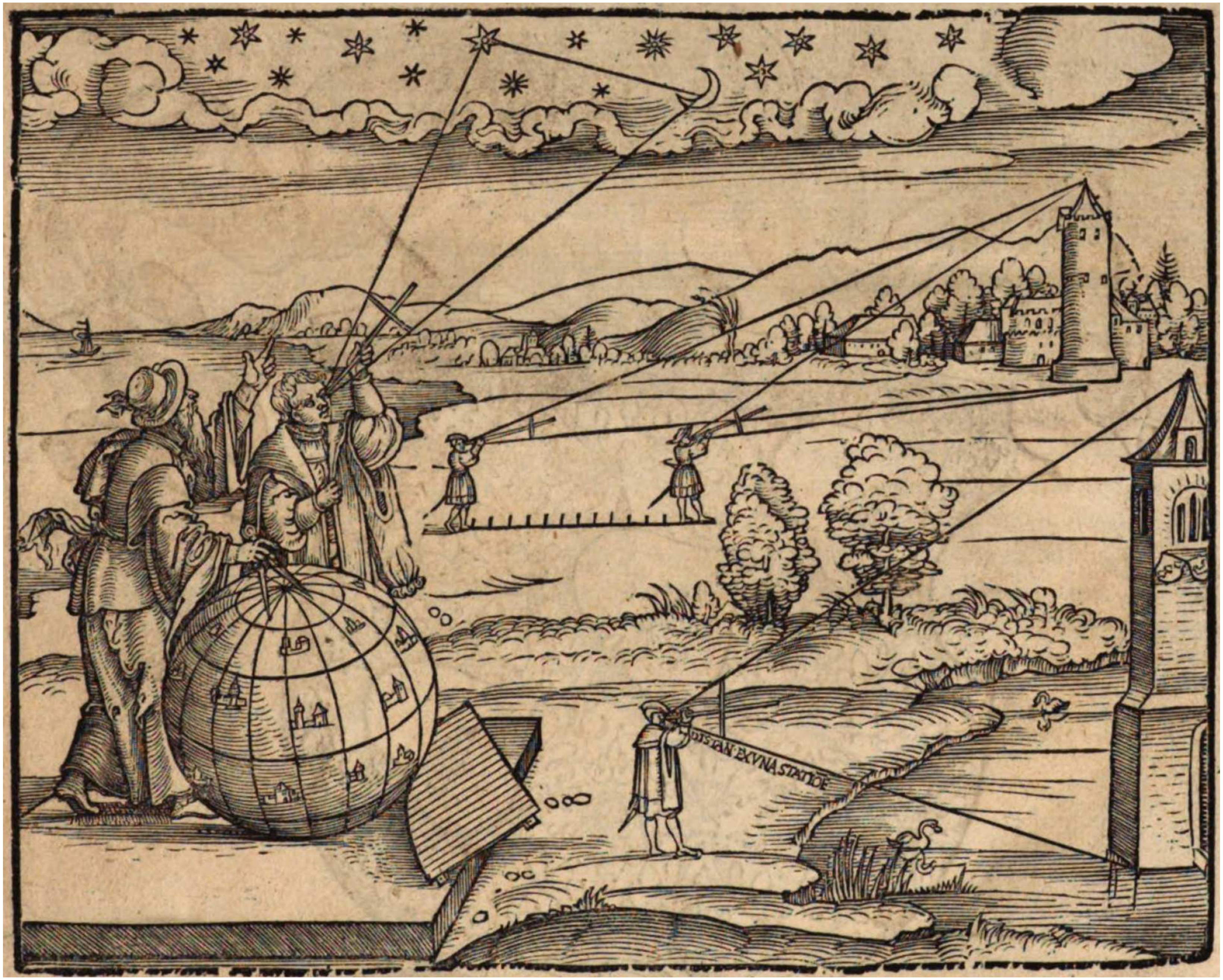
An Act for Providing a Publick Reward for ſuch Perſon or Perſons as ſhall Diſcover the Longitude at Sea.

Whereas it is well known by all that are acquainted with the Art of Navigation, That nothing is ſo much wanted and deſired at Sea, as the Diſcovery of the Longitude, for the Safety and Quickneſs of Voyages, the Preſervation of Ships and the Lives of Men: And whereas in the Judgment of Able Mathematicians and Navigators, ſeveral Methods have already been Diſcovered, true in Theory, though very Diſficult in Practice, ſome of which (there is reaſon to expect) may be capable of Improvement, ſome already Diſcovered may be propoſed to the Publick, and others may be Invented hereafter: And whereas ſuch a Diſcovery would be of particular Advantage to the Trade of

“Methods have already been Discovered, true in Theory”

Johann Werner described how to determine longitude using the position of the moon in 1514.

Petrus Apianus discussed how to measure the position of the moon in 1533.


DISTAN·EX VNA STATIOE

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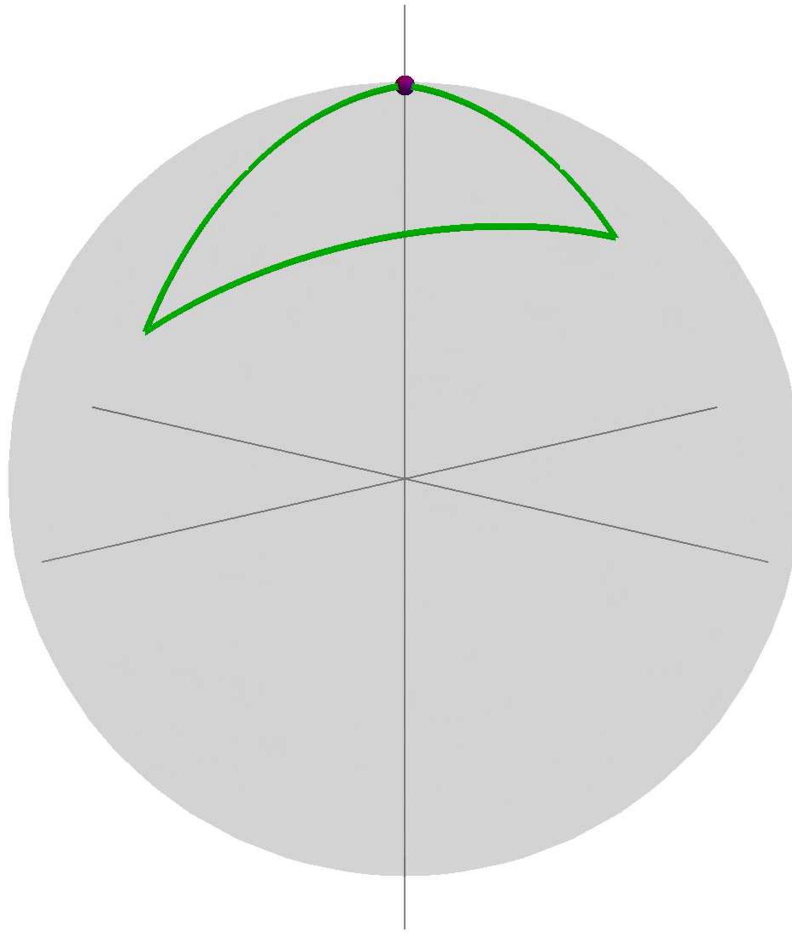
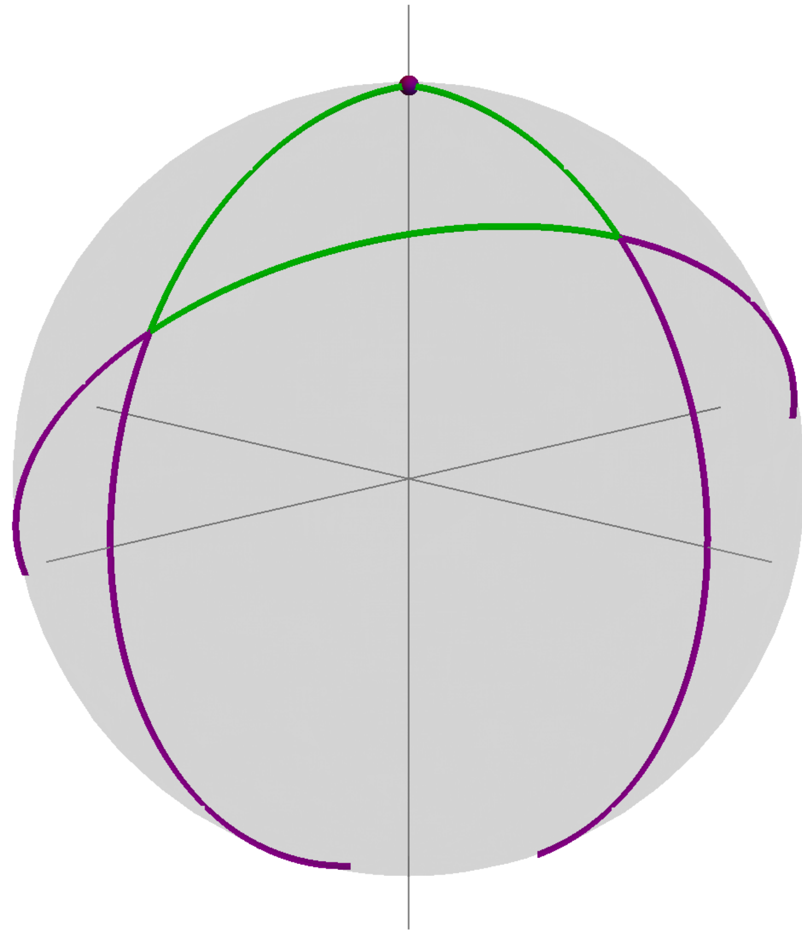
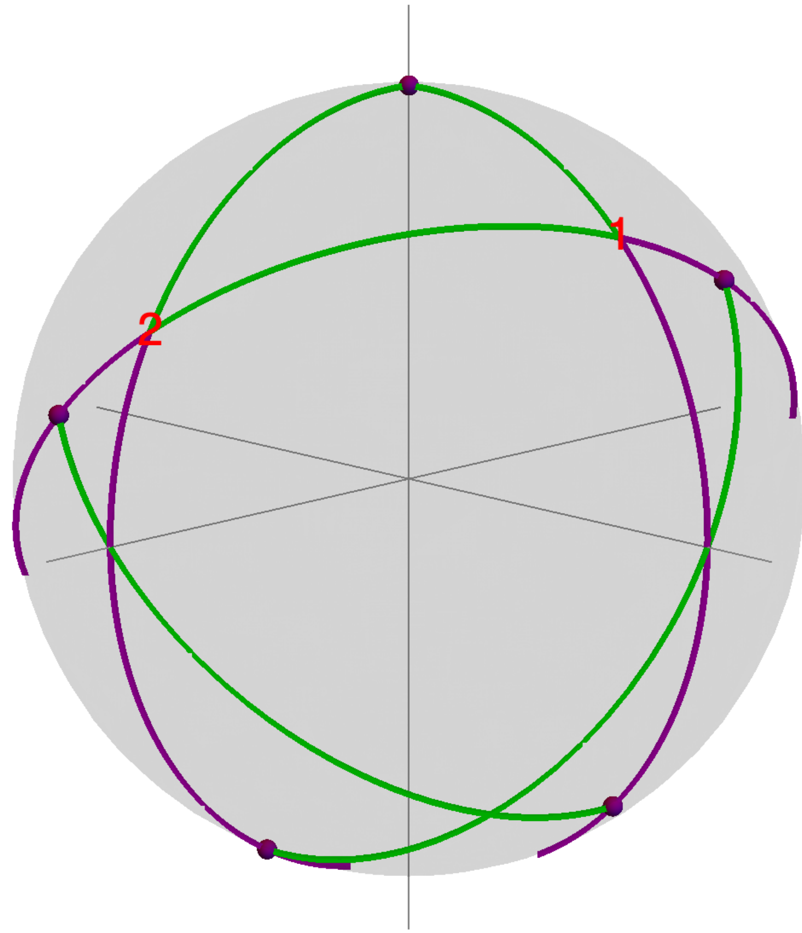
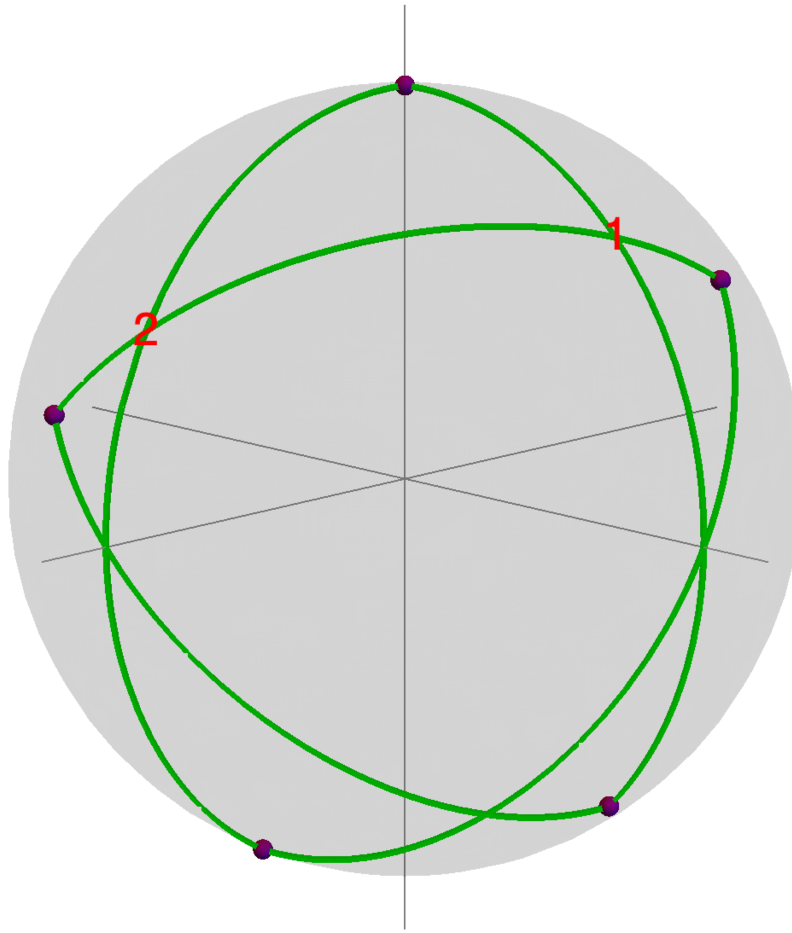
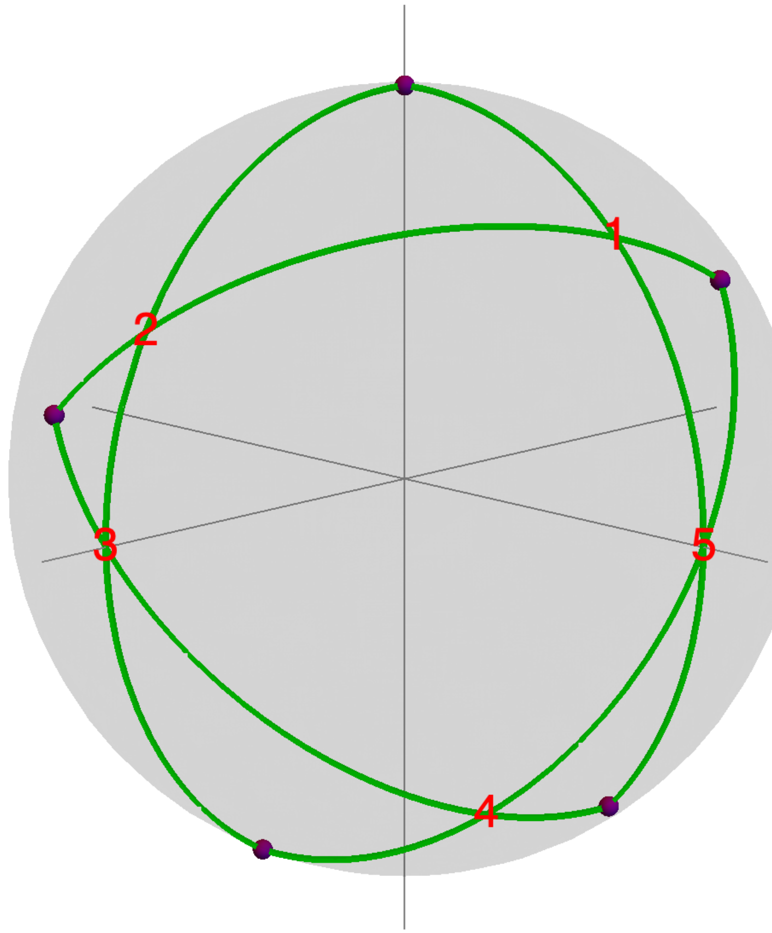
He also proved a remarkable theorem.

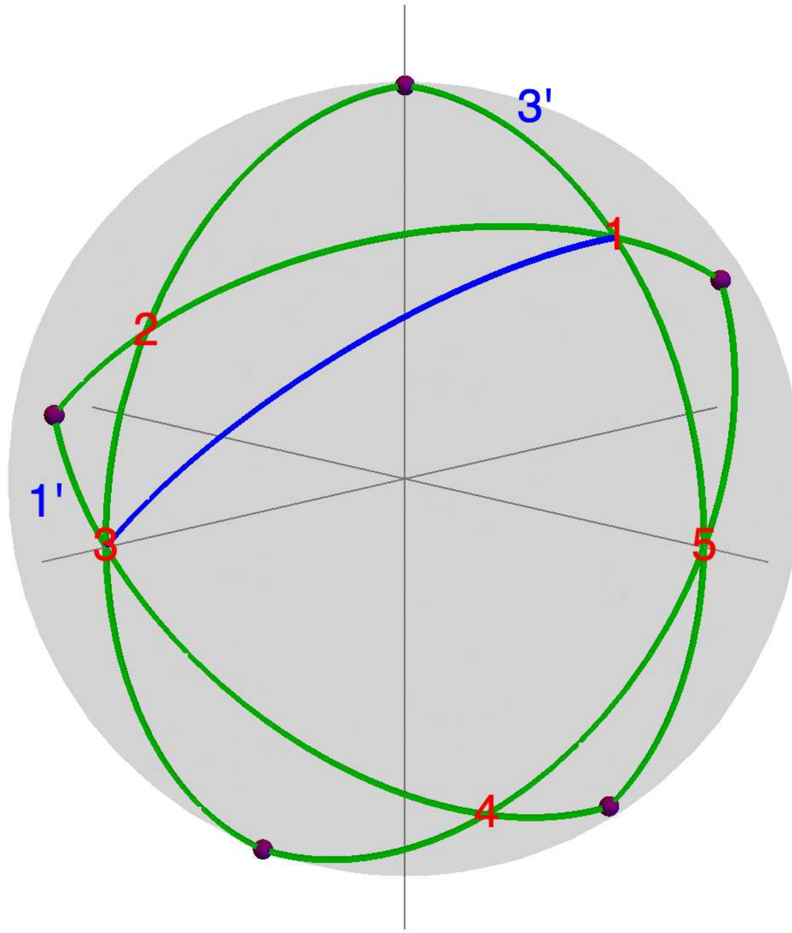




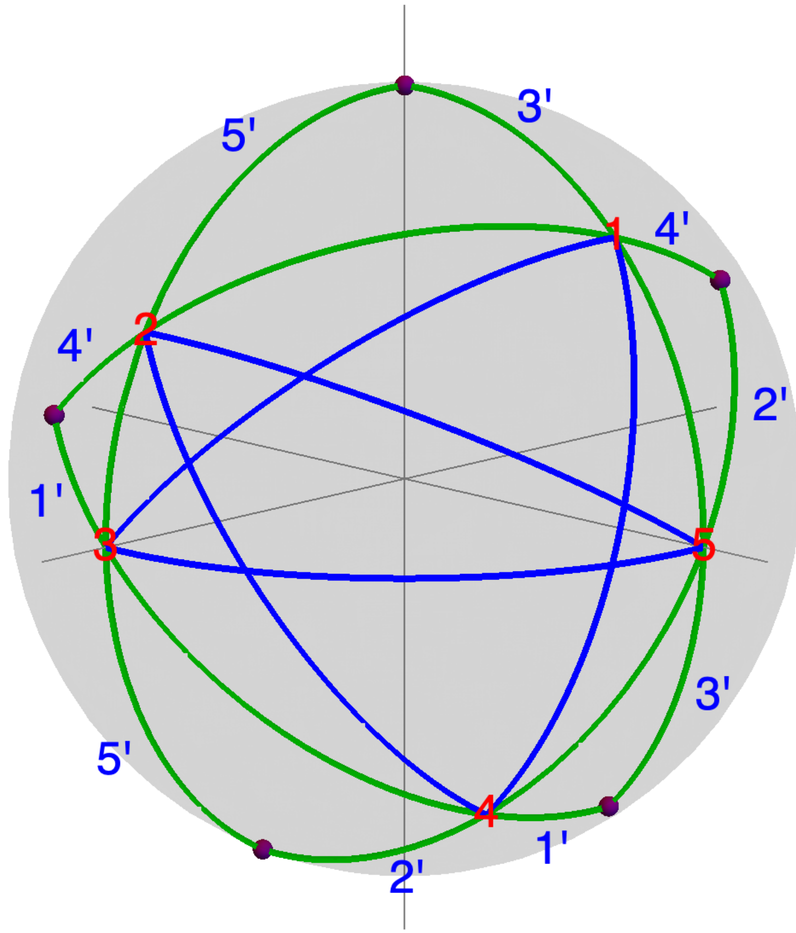

1
2


1
2

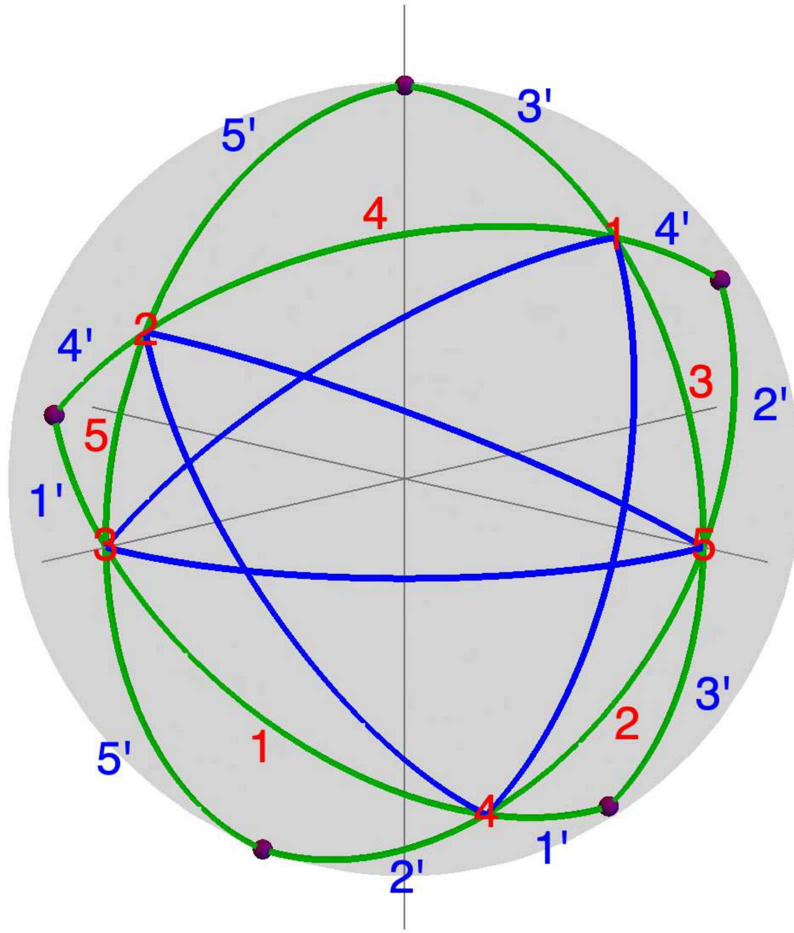

1
2
3
4
5



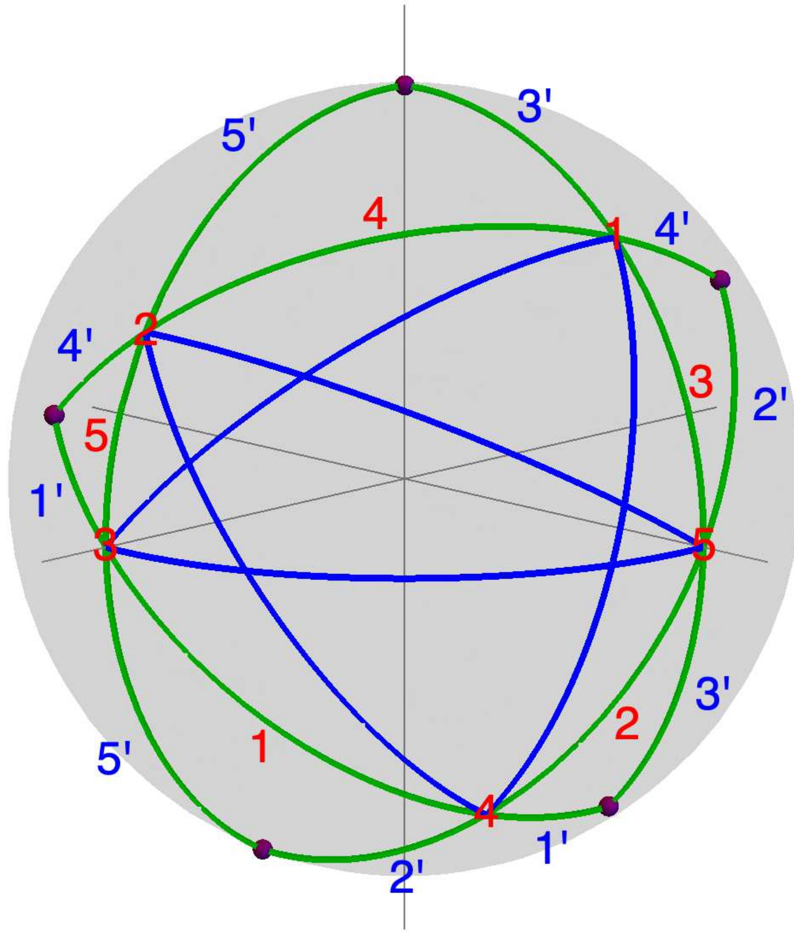
3'
1
2
1'
3
5
4



5'
3'
1
4'
2
4'
2'
1'
3
5
3'
5'
4
1'
2'



3'
5'
4
1
4'
2
4'
3
2'
5
1'
3
5
3'
2
5'
1
4
1'
2'



Circular parts: $5'3'1'4'2'$, $1'4'2'5'3'$, $2'5'3'1'4'$, $3'1'4'2'5'$, $3'4'2'5''1'$.

$$\cos 4 = \cos 5' \cos 3' \Rightarrow \sin 4' = \cos 5' \cos 3'$$

$$\cos 1 = \tan 3' \cot 4 \ \Rightarrow \sin 1' = \tan 3' \tan 4'$$

Theorems

Let $(p'_i, p'_{i+3}, p'_{i+1}, p'_{i+4}, p'_{i+2})$ be the circular parts of a right spherical triangle, where the subscripts are interpreted modulo 5.

Napier (1614). $\sin p'_i = \cos p'_{i+1} \cos p'_{i-1} = \tan p'_{i+2} \tan p'_{i-2}$.

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Gauß (1876). Let $\alpha_i = \tan^2 p_i$. Then $\alpha_i + 1 = \alpha_{i+2}\alpha_{i+3}$.

Proof. [exercise]

A frieze pattern

$$
\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\alpha_5 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_1 \\
& \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_1 & & \alpha_2 & & \alpha_3 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

A frieze pattern

$$\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\alpha_5 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_1 \\
& \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_1 & & \alpha_2 & & \alpha_3 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Find a solution with $\alpha_i \in \mathbb{N}$.

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