Assigned Problems (write up full solutions and hand in):

Section 3.2 #24
Section 3.3. #5, 6
Section 3.4 #2, 4, 6

Problems not from the text (also to be handed in)

A. Let $G$ be a group and let $a$ and $b$ be elements of $G$ which have finite order. For convenience, define $m = o(a) < \infty$ and $n = o(b) < \infty$. Suppose that $a$ and $b$ commute, that is, $ab = ba$. Let $k = o(ab)$ be the order of the element $ab$. The point of this problem is to study how the order of $ab$ is related to the order of $a$ and the order of $b$.

(a). Prove that $k$ is finite and in fact that $k$ divides $\text{lcm}(m, n)$.

(b). Show that if $\gcd(m, n) = 1$, then $k = mn = \text{lcm}(m, n)$. 
(c). Give an example showing that $k$ can be smaller than $\text{lcm}(m, n)$ in general.

B. If two elements $a$ and $b$ of finite order do not commute, the result of the previous exercise fails completely; it is even possible for $ab$ to have infinite order. In this exercise you see an example of this.

Let $G$ be the group of all permutations of the infinite set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \}$. Remember that $G$ is the set of all bijective functions $f : \mathbb{Z} \to \mathbb{Z}$ and that the operation in $G$ is composition of functions, i.e. $fg$ means $f \circ g$. Let $f, g \in G$ be the functions given by the formulas $f(x) = -x$ and $g(x) = 1 - x$ (you can take as given that these really are bijective functions and thus belong to $G$.) Prove that in the group $G$, $o(f) = 2$ and $o(g) = 2$, but that $o(fg) = \infty$.

C. In any group $G$, if $H$ and $K$ are subsets of $G$ then we define $HK = \{hk | h \in H, k \in K \}$.

(a). Prove that if $H$ and $K$ are subgroups of $G$, then $HK$ is a subgroup of $G$ if and only if $HK = KH$. (do not use Proposition 3.3.2 in the text).

(b). Use the result of (a) to give an alternative proof of Proposition 3.3.2 in the text: namely, if $H$ and $K$ are subgroups of $G$ and $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, then $HK$ is a subgroup of $G$.

Optional problems (handing in not required)

(None this week)