Math 103A Fall 2007 Exam 1

October 31, 2007

NAME: Solutions

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Problem 1 (30 points)

1. Let $D_3 = \{R_0, R_{120}, R_{240}, S_1, S_2, S_3\}$ be the dihedral group of order 6, which consists of symmetries of an equilateral triangle. Here, each $R_i$ is the symmetry of the square given by counterclockwise rotation by $i$ degrees. Each $S_i$ is a reflection about an axis of symmetry of the triangle, labeled as follows.

   (As in class, orient the triangle with horizontal base, let the top corner have label 1, the bottom left corner have label 2, and the bottom right corner have label 3. Then $S_i$ is the reflection through the axis of symmetry passing through the corner $i$.)

(1a) (10 pts). Calculate the product $S_1S_2$ in the group $D_3$. Show your work.

   $S_1S_2 = R_{120}$.

   (A common error was to first perform $S_1$ and then $S_2$. The convention in $D_3$ is to multiply from right to left, as we do with function composition.)
(1b) (10 pts). Complete the following Cayley table of the group $D_3$. (Remember that the product $g_1g_2$ is written in row $g_1$ and column $g_2$ of the Cayley table.) You may freely rely on facts you know about Cayley tables.

(The filled in entries are in italics. Once one fills in $S_1S_2 = R_{120}$, the rest of the table follows from the fact that every row and column contains each element exactly once.)

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(1c) (10 pts). Calculate $Z(D_3)$, in other words, find the center of the group $D_3$. Justify your answer.

$Z(D_3) = \{ x \in D_3 \mid xa = ax \text{ for all } a \in D_3 \}$. In terms of the Cayley table, an element $x$ is in the center if and only if row $x$ and column $x$ are identical. Looking at the Cayley table above, we see that this happens only when $x = R_0$. So $Z(D_3) = \{ R_0 \}$. 

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Problem 2 (25 points)

(a) \((15 \text{ pts})\) Let \(G\) be an Abelian group with identity element \(e\), and define \(H = \{x \in G \mid x^2 = e\}\). Prove that \(H\) is a subgroup of \(G\).

We use the two-step subgroup test. Let \(a \in H\) and \(b \in H\). Then \(a^2 = e\) and \(b^2 = e\). Now \((ab)^2 = abab = a^2b^2 = ee = e\) since \(G\) is Abelian. Thus \((ab) \in H\). So \(H\) is closed under products.

Next, note that \((a^{-1})^2 = (a^2)^{-1} = e^{-1} = e\). So \(a^{-1} \in H\) and \(H\) is also closed under inverses. By the two step subgroup test, \(H\) is a subgroup of \(G\).
(b) (10 pts) Let $G = D_3$ be the group of symmetries of a triangle (the group appearing in problem 1), and again define $H = \{x \in G \mid x^2 = e\}$. Prove that $H$ is not a subgroup of $G$.

From the Cayley table of problem 1, we see that $R_0, S_1, S_2$ and $S_3$ are exactly the elements of $D_3$ whose square is the identity. However, $H = \{R_0, S_1, S_2, S_3\}$ is not a subgroup of $G$, because it is not closed under products for example, $S_1S_2 = R_{120} \notin H$. 
Problem 3 (25 points)

(a) (10 pts). Show that the group $U(14)$ is a cyclic group. Show your calculations.

Note that $U(14) = \{[1], [3], [5], [9], [11], [13]\}$. One finds a generator by trial and error; it turns out that $[3]$ works, because


so $[3]$ has order 6 and $\langle [3] \rangle = \{[1], [3], [9], [13], [11], [5] \} = U(14)$. 
(b) (5 pts). Find a subgroup $H$ of $U(14)$ with $|H| = 3$.

By the theory of cyclic groups, since $[3]$ has order 6, $[3]^2 = [9]$ will have order 3. So $H = \langle [9] \rangle = \{1, [9], 11\}$ works (and in fact this is the only answer.)

(d) (10 pts). Working in the group $U(14)$ still, calculate $[9]^{-100}$. Show your work.

Problem 4 (20 points)

(a) (10 pts). Let $S = \mathbb{R}$ be the set of real numbers, and suppose we define a binary operation on $S$ by the formula $a \ast b = a - b$. Is $S$ with this binary operation a group? Either prove it is a group or prove it is not a group.

Since $1 = 1 - (1 - 1) \neq (1 - 1) - 1 = -1$, the operation is not associative, so $S$ with this operation is not a group.

(The operation also does not have an identity element: If it did, there would be some number $e$ such that $x = x - e = e - x$, for all $x \in \mathbb{R}$, but no such number $e$ exists. So you could also prove it is not a group that way.)
(b) (10 pts). Let $S = \{x \in \mathbb{R} \mid x \neq 0\}$ be the set of nonzero real numbers, and define a a binary operation on $S$ by the formula $a \star b = 2ab$. Is $S$ with this binary operation a group? Either prove it is a group or prove it is not a group.

This is a group. It is associative, since $(a \star b) \star c = 2ab \star c = 4abc$ and $a \star (b \star c) = a \star 2bc = 4abc$, for all $a, b, c \in \mathbb{R}$ (where we have assumed that usual multiplication of real numbers is associative.)

It has an identity element $e = 1/2$, since $a \star (1/2) = a = (1/2) \star a$ for all $a \in \mathbb{R}$.

It has inverses, since for $a \in S$, we see that $b = 1/(4a)$ is an inverse: $a \star 1/(4a) = 1/2 = 1/(4a) \star a$.

So $S$ with the given operation is a group by definition.