Math 103A Fall 2007 Exam 1

October 31, 2007

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Problem 1 (30 points)

1 Let $D_3 = \{R_0, R_{120}, R_{240}, S_1, S_2, S_3\}$ be the dihedral group of order 6, which consists of symmetries of an equilateral triangle. Here, each R_i is the symmetry of the square given by *counterclockwise* rotation by i degrees. Each S_i is a reflection about an axis of symmetry of the triangle, labeled as follows.

(As in class, orient the triangle with horizontal base, let the top corner have label 1, the bottom left corner have label 2, and the bottom right corner have label 3. Then S_i is the reflection through the axis of symmetry passing through the corner i.)

(1a) (10 pts). Calculate the product $S_1 S_2$ in the group D_3. Show your work.

$S_1 S_2 = R_{120}$.

(A common error was to first perform S_1 and then S_2. The convention in D_3 is to multiply from right to left, as we do with function composition.)

(1b) (10 pts). Complete the following Cayley table of the group D_3. (Remember that the product $g_1 g_2$ is written in row g_1 and column g_2 of the Cayley table.) You may freely rely on facts you know about Cayley tables.

(The filled in entries are in italics. Once one fills in $S_1 S_2 = R_{120}$, the rest of the table follows from the fact that every row and column contains each element exactly once.)

	$\mathbf{R_0}$	$\mathbf{R_{120}}$	$\mathbf{R_{240}}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$
$\mathbf{R_0}$	R_0	R_{120}	R_{240}	S_1	S_2	S_3
$\mathbf{R_{120}}$	R_{120}	R_{240}	R_0	S_3	S_1	S_2
$\mathbf{R_{240}}$	R_{240}	R_0	R_{120}	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3	R_{0}	R_{120}	R_{240}
$\mathbf{S_2}$	S_2	S_3	S_1	R_{240}	R_{0}	R_{120}
$\mathbf{S_3}$	S_3	S_1	S_2	R_{120}	R_{240}	R_{0}

(1c) (10 pts). Calculate $Z(D_3)$, in other words, find the *center* of the group D_3. Justify your answer.

$Z(D_3) = \{x \in D_3 | \; xa = ax \text{ for all } a \in D_3\}$. In terms of the Cayley table, an element x is in the center if and only if row x and column x are identical. Looking at the Cayley table above, we see that this happens only when $x = R_0$. So $Z(D_3) = \{R_0\}$.

Problem 2 (25 points)

(a) (15 pts) Let G be an *Abelian* group with identity element e, and define $H = \{x \in G \,|\, x^2 = e\}$. Prove that H is a subgroup of G.

We use the two-step subgroup test. Let $a \in H$ and $b \in H$. Then $a^2 = e$ and $b^2 = e$. Now $(ab)^2 = abab = a^2b^2 = ee = e$ since G is Abelian. Thus $(ab) \in H$. So H is closed under products.

Next, note that $(a^{-1})^2 = (a^2)^{-1} = e^{-1} = e$. So $a^{-1} \in H$ and H is also closed under inverses. By the two step subgroup test, H is a subgroup of G.

(b) (10 pts) Let $G = D_3$ be the group of symmetries of a triangle (the group appearing in problem 1), and again define $H = \{x \in G \mid x^2 = e\}$. Prove that H is *not* a subgroup of G.

From the Cayley table of problem 1, we see that R_0, S_1, S_2 and S_3 are exactly the elements of D_3 whose square is the identity. However, $H = \{R_0, S_1, S_2, S_3\}$ is not a subgroup of G, because it is not closed under products—for example, $S_1 S_2 = R_{120} \notin H$.

Problem 3 (25 points)

(a) (10 pts). Show that the group $U(14)$ is a cyclic group. Show your calculations.

Note that $U(14) = \{[1], [3], [5], [9], [11], [13]\}$. One finds a generator by trial and error; it turns out that [3] works, because

$$[3]^1 = [3], [3]^2 = [9], [3]^3 = [13], [3]^4 = [11], [3]^5 = [5], \text{and } [3]^6 = [1],$$

so [3] has order 6 and $\langle[3]\rangle = \{[1], [3], [9], [13], [11], [5]\} = U(14)$.

(b) (5 pts). Find a subgroup H of $U(14)$ with $|H| = 3$.

By the theory of cyclic groups, since $[3]$ has order 6, $[3]^2 = [9]$ will have order 3. So $H = \langle [9] \rangle = \{[1], [9], 11]\}$ works (and in fact this is the only answer.)

(d) (10 pts). Working in the group $U(14)$ still, calculate $[9]^{-100}$. Show your work.

Since $[9]$ has order 3 in the group $U(14)$, we have $[9]^{-100} = [9]^2$ (since -100 is congruent to 2 mod 3) and $[9]^2 = [11]$. So $[9]^{-100} = [11]$.

Problem 4 (20 points)

(a) (10 pts). Let $S = \mathbb{R}$ be the set of real numbers, and suppose we define a binary operation on S by the formula $a \star b = a - b$. Is S with this binary operation a group? Either prove it is a group or prove it is not a group.

Since $1 = 1 - (1 - 1) \neq (1 - 1) - 1 = -1$, the operation is not associative, so S with this operation is not a group.

(The operation also does not have an identity element: If it did, there would be some number e such that $x = x - e = e - x$, for all $x \in \mathbb{R}$, but no such number e exists. So you could also prove it is not a group that way.)

(b) (10 pts). Let $S = \{x \in \mathbb{R} | x \neq 0\}$ be the set of nonzero real numbers, and define a a binary operation on S by the formula $a \star b = 2ab$. Is S with this binary operation a group? Either prove it is a group or prove it is not a group.

This is a group. It is associative, since $(a \star b) \star c = 2ab \star c = 4abc$ and $a \star (b \star c) = a \star 2bc = 4abc$, for all $a, b, c \in \mathbb{R}$ (where we have assumed that usual multiplication of real numbers is associative.)

It has an identity element $e = 1/2$, since $a \star (1/2) = a = (1/2) \star a$ for all $a \in \mathbb{R}$.

It has inverses, since for $a \in S$, we see that $b = 1/(4a)$ is an inverse: $a \star 1/(4a) = 1/2 = 1/(4a) \star a$.

So S with the given operation is a group by definition.