

# Math 103A Fall 2005 Exam 2

November 9, 2005

NAME:

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## Problem 1 (30 points)

1a (10 pts) List all of the subgroups of the cyclic group  $\mathbb{Z}_{27}$ .

1b (10 pts) How many elements of order 3 are there in the group  $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ ?

**1c (10 pts)** Show that  $\mathbb{Z}_{27}$  is not isomorphic to  $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ .

## Problem 2 (20 points)

**2a (10 pts)** Write the permutation  $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 1 & 2 & 7 & 6 & 8 & 3 \end{bmatrix}$  in disjoint cycle form.

**2b (5 pts)** Write  $\alpha$  as a product of transpositions (2-cycles). Is  $\alpha \in A_8$ ?

**2c (5 pts)** What is the order of  $\alpha$ ?

### Problem 3 (15 points)

**2d (10 pts)** Consider the group  $S_4$  and let  $\beta = (1234) \in S_4$ . Show that  $\beta$  is not in the center of  $S_4$ .

**2e (5 pts)** Let  $H = \langle \beta \rangle$  be the cyclic subgroup of  $S_4$  generated by  $\beta$ . How many distinct left cosets of  $H$  in  $S_4$  are there? (Your answer should be an actual number and should not involve symbols.)

## Problem 4 (20 points)

In parts (a)-(c) of this problem, you may use without proof the following formulas for the structure of  $U(n)$  when  $n$  is a prime power:  $U(2) \cong \{e\}$ ,  $U(4) \cong \mathbb{Z}_2$ ,  $U(2^m) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{m-2}}$  when  $m \geq 3$ ,  $U(p^m) \cong \mathbb{Z}_{p^m - p^{m-1}}$  when  $p$  is an odd prime and  $m \geq 1$ .

**4a (5 pts)** Show that  $U(55)$  is isomorphic to a direct product of cyclic groups.

**4b (5 pts)** Show that  $U(75)$  is isomorphic to a direct product of cyclic groups.

**4c (5 pts)** Show that  $U(55)$  and  $U(75)$  are isomorphic to each other.

**4d (5 pts)** It is claimed in the formulas on the previous page that  $U(4)$  is isomorphic to  $\mathbb{Z}_2$ . Explain why this must be true.

## Problem 5 (15 points)

**5a (10 pts)** Suppose that  $G$  is a nonabelian group with  $|G| = 14$ , and that  $x \in G$  is an element such that  $x^7 \neq e$ . Find  $|x|$ .

**5b (5 pts)** Explain why (i.e. perhaps by quoting a theorem)  $G$  must be isomorphic to the dihedral group  $D_7$ .