Math 103A Fall 2006 Exam 2

November 8, 2006

Exam with Solutions

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Problem 1 (25 pts)

In this problem, let S_n be the symmetric group of degree n, in other words the group of permutations of the set $\{1, 2, \ldots, n\}$, and let A_n be the alternating subgroup.

1a (5 pts). Write the element $\alpha = (1523)(3574) \in S_7$ as a product of *disjoint* cycles.

By direct calculation we find $\alpha = (1574)(23)$.

1b (10 pts). Find an element $\beta \in A_7$ (that is, an *even* permutation) such that $\text{order}(\beta) = 4$. Explain how you know that your β is even and has the correct order.

One possibility is $\beta = (1234)(56)$. By the theorem on the order of a permutation in disjoint cycle form, we have $\text{order}(\beta) = \text{lcm}(2, 4) = 4$. Also, $\beta = (14)(13)(12)(56)$ is a product of 4 transpositions, so it is an even permutation.

(In fact basically the only solution to this problem is the product of a 4-cycle and a 2-cycle which are disjoint, although it doesn't matter exactly what numbers occur in the cycles.)

1c (10 pts). Prove that S_3 is non-Abelian.

We calculate that $(12)(13) = (132)$ but $(13)(12) = (123)$. Also note that (123) and (132) are different permutations since the first sends 1 to 2 but the second sends 1 to 3. Thus (12) and (13) do not commute in S_3 and so S_3 is not an Abelian group.

Problem 2 (25 pts)

2a (15 pts). Show that [3] is a generator for the group $U(7)$. Then write down the list of all possible subgroups of $U(7)$. Indicate the order of each subgroup you found.

Note that $|U(7)| = 6$. We calculate $[3]^1 = [3]$, $[3]^2 = [9] = [2]$, $[3]^3 = [2][3] = [6]$, $[3]^4 = [3][6] = [18] = [4]$, $[3]^5 = [3][4] = [12] = [5]$, and $[3]^6 = [1]$. Thus $\langle [3] \rangle = U(7)$.

In particular, $U(7)$ is cyclic. Thus we can apply the theorem on subgroups of cyclic groups to find all of the subgroups of $U(7)$. They are

$$\langle [3]^1 \rangle = \{[3]^0, [3]^1, [3]^2, [3]^3, [3]^4, [3]^5\} = U(7) \quad \text{(order 6)},$$

$$\langle [3]^2 \rangle = \{[3]^0, [3]^2, [3]^4\} = \{[1], [2], [4]\} \quad \text{(order 3)},$$

$$\langle [3]^3 \rangle = \{[3]^0, [3]^3\} = \{[1], [6]\} \quad \text{(order 2)},$$

$$\langle [3]^6 \rangle = \{[3]^0\} = \{[1]\} \quad \text{(order 1)}.$$

2b (10 pts). Find the order of the element [8] in the group $\mathbb{Z}_{90}$.

There is a formula for the order, but we just calculate it directly. The order of [8] will be the smallest $n > 0$ such that $n[8] = [8n] = [0]$ in $\mathbb{Z}_{90}$, in other words the smallest $n > 0$ such that $8n$ is a multiple of 90. The easiest way to find this is to check the multiples of 90 until you find one that is also multiple of 8: the smallest is 360. So the n you want is the n such that $8n = 360$, namely $n = 45$. This answer agrees with the formulas I gave in class, which give $n = \operatorname{lcm}(8, 90)/8 = 90/\gcd(8, 90) = 45$.

Problem 3 (25 pts)

In this problem, Let G be the group $U(7)$, and let H be the subgroup $H = \langle[6]\rangle = \{[1],[6]\}$ of G.

3a (5 pts). Explain *briefly* how you know that the factor group G/H is well-defined (this is not supposed to take any work.)

The factor group G/H may be defined as long as H is a normal subgroup of G. Since G is Abelian, every subgroup of G is normal, so G/H is well-defined.

3b (10 pts). Find the order of $[3]H$ in the group G/H. Show your work.

We calculate directly that $([3]H)^1 = [3]H$, $([3]H)^2 = [9]H = [2]H$, and $([3]H)^3 = [6]H$ (We have already done the work for this calculation in Problem 2.) Note that $[6]H = H = [1]H$ since $[6] \in H$, so $([3]H)^3 = [1]H$ is the identity element in G/H. On the other hand, $[3]H \neq H$ because $[3] \notin H$, and $[2]H \neq H$ because $[2] \notin H$. So 3 is the smallest positive n such that $([3]H)^n = [1]H$, and thus order$([3]H) = 3$.

3c (10 pts). Prove that G/H is isomorphic to $\mathbb{Z}_3$. You may use theorems we proved in class but make sure you say what results you are using.

Note that $|G/H| = |G|/|H| = 3$. Since $[3]H$ has order 3 in G/H as we have just shown, that element generates G/H and so G/H is a cyclic group. (Alternatively, you had a homework exercise that proved that any factor group of a cyclic group is cyclic, and G is cyclic by problem 2.) Now we proved in class that all cyclic groups of order n are isomorphic to $\mathbb{Z}_n$. So $G/H \cong \mathbb{Z}_3$.

Problem 4 (25 points)

4a (10 pts). There exists a homomorphism $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ with the property that $\phi([1]) = [3]$. Find the homomorphism ϕ (for example by giving a formula for it). Then explain (1) why your answer is the only possibility; and (2) why your answer really does give a well-defined homomorphism.

One of the basic properties of homomorphisms is that $\phi(x^n) = \phi(x)^n$ for all $n \in \mathbb{Z}$. Since both the domain and target group in this example are additive groups, this translates to the statement that $\phi(n \cdot [x]) = n \cdot \phi([x])$ for all $[x] \in \mathbb{Z}_{12}$. Applying this with $[x] = [1]$, we see that $\phi([n]) = \phi(n \cdot [1]) = n \cdot \phi([1]) = n[3] = [3n]$ for all $n \in \mathbb{Z}$. Thus $\phi([n]) = [3n]$ is the required formula for ϕ. We have also shown that this formula is forced by the facts that $\phi([1]) = [3]$ and ϕ is a homomorphism, so our answer is the only possibility.

To show that our formula really does define a homomorphism, first we must show that ϕ is a well-defined function. For this we need to check that if $m, n \in \mathbb{Z}$ and $[m] = [n]$, then $[3n] = [3m]$. But this is clear, since if $[m] = [n]$ then $12|(m - n)$, so certainly $12|(3m - 3n)$ and thus $[3n] = [3m]$. Once we know that ϕ is a well-defined function, we can easily prove it satisfies the rule for a homomorphism: $\phi([m] + [n]) = \phi([m + n]) = [3m + 3n] = [3m] + [3n] = \phi([m]) + \phi([n])$.

4b (10 pts). Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be the same homomorphism as in part (a). Find (1) the kernel $\ker \phi$ of the homomorphism; (2) the image $\phi(\mathbb{Z}_{12})$ of the homomorphism.

The kernel is all $[x] \in \mathbb{Z}_{12}$ such that $\phi([x]) = [3x] = [0] \in \mathbb{Z}_{12}$. But $[3x] = [0]$ if and only if $12|(3x)$, which happens if and only if $4|x$. So $\ker \phi = \{[0], [4], [8]\}$.

The image is $\{[3x]|x \in \mathbb{Z}\} = \langle [3] \rangle = \{[0], [3], [6], [9]\}$.

4c (5 pts). (This part has nothing to do with parts (a) and (b)). Let $\phi : \mathbb{Z}_9 \to D_7$ be a homomorphism, where D_7 is the dihedral group of order 14. Find, with proof, ϕ. State any theorems you use.

There are several ways to do this problem; here is one using orders of elements. Assume ϕ is a homomorphism. Since $[1]$ has order 9 in $\mathbb{Z}_9$, one of the basic properties of homomorphisms states that $\phi([1])$ has order dividing 9 in D_7. Since $|D_7| = 14$, by Lagrange's theorem $\phi([1])$ also has order dividing 14. This forces $\phi([1])$ to have order 1, so $\phi([1]) = e = R_0$. But now by the properties of homomorphisms we also get that $\phi(n \cdot [1]) = (R_0)^n$ for all $n \geq 0$, so $\phi([n]) = R_0$ for all n. Thus ϕ is the trivial homomorphism, in other words the homomorphism sending every element of $\mathbb{Z}_9$ to the identity of D_7.