Math 103A Fall 2006 Exam 2

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Exam with Solutions

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Problem 1 (25 pts)

In this problem, let $S_n$ be the symmetric group of degree $n$, in other words the group of permutations of the set $\{1, 2, \ldots, n\}$, and let $A_n$ be the alternating subgroup.

1a (5 pts). Write the element $\alpha = (123)(3574) \in S_7$ as a product of disjoint cycles.

By direct calculation we find $\alpha = (1574)(23)$.

1b (10 pts). Find an element $\beta \in A_7$ (that is, an even permutation) such that $\text{order}(\beta) = 4$. Explain how you know that your $\beta$ is even and has the correct order.

One possibility is $\beta = (1234)(56)$. By the theorem on the order of a permutation in disjoint cycle form, we have $\text{order}(\beta) = \text{lcm}(2, 4) = 4$. Also, $\beta = (14)(13)(12)(56)$ is a product of 4 transpositions, so it is an even permutation.

(In fact basically the only solution to this problem is the product of a 4-cycle and a 2-cycle which are disjoint, although it doesn’t matter exactly what numbers occur in the cycles.)
1c (10 pts). Prove that $S_3$ is non-Abelian.

We calculate that $(12)(13) = (132)$ but $(13)(12) = (123)$. Also note that $(123)$ and $(132)$ are different permutations since the first sends 1 to 2 but the second sends 1 to 3. Thus $(12)$ and $(13)$ do not commute in $S_3$ and so $S_3$ is not an Abelian group.
Problem 2 (25 pts)

2a (15 pts). Show that $[3]$ is a generator for the group $U(7)$. Then write down the list of all possible subgroups of $U(7)$. Indicate the order of each subgroup you found.


In particular, $U(7)$ is cyclic. Thus we can apply the theorem on subgroups of cyclic groups to find all of the subgroups of $U(7)$. They are


\[ \langle [3]^6 \rangle = \{ [3]^0 \} = \{ [1] \} \quad \text{(order 1)}. \]
2b (10 pts). Find the order of the element $[8]$ in the group $\mathbb{Z}_{90}$.

There is a formula for the order, but we just calculate it directly. The order of $[8]$ will be the smallest $n > 0$ such that $n[8] = [8n] = [0]$ in $\mathbb{Z}_{90}$, in other words the smallest $n > 0$ such that $8n$ is a multiple of 90. The easiest way to find this is to check the multiples of 90 until you find one that is also multiple of 8: the smallest is 360. So the $n$ you want is the $n$ such that $8n = 360$, namely $n = 45$. This answer agrees with the formulas I gave in class, which give $n = \text{lcm}(8, 90)/8 = 90/\gcd(8, 90) = 45$. 
Problem 3 (25 pts)

In this problem, Let $G$ be the group $U(7)$, and let $H$ be the subgroup $H = \langle [6] \rangle = \{[1], [6]\}$ of $G$.

3a (5 pts). Explain briefly how you know that the factor group $G/H$ is well-defined (this is not supposed to take any work.)

The factor group $G/H$ may be defined as long as $H$ is a normal subgroup of $G$. Since $G$ is Abelian, every subgroup of $G$ is normal, so $G/H$ is well-defined.

3b (10 pts). Find the order of $[3]H$ in the group $G/H$. Show your work.

3c (10 pts). Prove that $G/H$ is isomorphic to $\mathbb{Z}_3$. You may use theorems we proved in class but make sure you say what results you are using.

Note that $|G/H| = |G|/|H| = 3$. Since $[3]H$ has order 3 in $G/H$ as we have just shown, that element generates $G/H$ and so $G/H$ is a cyclic group. (Alternatively, you had a homework exercise that proved that any factor group of a cyclic group is cyclic, and $G$ is cyclic by problem 2.) Now we proved in class that all cyclic groups of order $n$ are isomorphic to $\mathbb{Z}_n$. So $G/H \cong \mathbb{Z}_3$. 

Problem 4 (25 points)

4a (10 pts). There exists a homomorphism $\phi : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ with the property that $\phi([1]) = [3]$. Find the homomorphism $\phi$ (for example by giving a formula for it). Then explain (1) why your answer is the only possibility; and (2) why your answer really does give a well-defined homomorphism.

One of the basic properties of homomorphisms is that $\phi(x^n) = \phi(x)^n$ for all $n \in \mathbb{Z}$. Since both the domain and target group in this example are additive groups, this translates to the statement that $\phi(n \cdot [x]) = n \cdot \phi([x])$ for all $[x] \in \mathbb{Z}_{12}$. Applying this with $[x] = [1]$, we see that $\phi([n]) = \phi(n \cdot [1]) = n \cdot \phi([1]) = n \cdot [3] = [3n]$ for all $n \in \mathbb{Z}$. Thus $\phi([n]) = [3n]$ is the required formula for $\phi$. We have also shown that this formula is forced by the facts that $\phi([1]) = [3]$ and $\phi$ is a homomorphism, so our answer is the only possibility.

To show that our formula really does define a homomorphism, first we must show that $\phi$ is a well-defined function. For this we need to check that if $m, n \in \mathbb{Z}$ and $[m] = [n]$, then $[3m] = [3n]$. But this is clear, since if $[m] = [n]$ then $12 | (m - n)$, so certainly $12 | (3m - 3n)$ and thus $[3m] = [3n]$. Once we know that $\phi$ is a well-defined function, we can easily prove it satisfies the rule for a homomorphism: $\phi([m] + [n]) = \phi([m + n]) = [3m + 3n] = [3m] + [3n] = \phi([m]) + \phi([n])$. 

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4b (10 pts). Let \( \phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12} \) be the same homomorphism as in part (a). Find (1) the kernel \( \ker \phi \) of the homomorphism; (2) the image \( \phi(\mathbb{Z}_{12}) \) of the homomorphism.

The kernel is all \([x] \in \mathbb{Z}_{12}\) such that \(\phi([x]) = [3x] = [0] \in \mathbb{Z}_{12}\). But \([3x] = [0]\) if and only if \(12 \mid (3x)\), which happens if and only if \(4 \mid x\). So \(\ker \phi = \{[0], [4], [8]\}\).

The image is \([3x]x \in \mathbb{Z}\) = \([3]\) = \([0], [3], [6], [9]\)\).

4c (5 pts). (This part has nothing to do with parts (a) and (b)). Let \( \phi : \mathbb{Z}_9 \to D_7 \) be a homomorphism, where \( D_7 \) is the dihedral group of order 14. Find, with proof, \( \phi \). State any theorems you use.

There are several ways to do this problem; here is one using orders of elements. Assume \( \phi \) is a homomorphism. Since \([1]\) has order 9 in \( \mathbb{Z}_9 \), one of the basic properties of homomorphisms states that \( \phi([1]) \) has order dividing 9 in \( D_7 \). Since \(|D_7| = 14\), by Lagrange’s theorem \( \phi([1]) \) also has order dividing 14. This forces \( \phi([1]) \) to have order 1, so \( \phi([1]) = e = R_0 \). But now by the properties of homomorphisms we also get that \( \phi(n \cdot [1]) = (R_0)^n \) for all \( n \geq 0 \), so \( \phi([n]) = R_0 \) for all \( n \). Thus \( \phi \) is the trivial homomorphism, in other words the homomorphism sending every element of \( \mathbb{Z}_9 \) to the identity of \( D_7 \).