Part 1: Short answers (4 points each)

For each question, either find an example of a ring, element, etc. with the requested properties (no explanation necessary); or else if no such example exists, give a one sentence explanation of why it is impossible.

Sample problems:

0a. A ring $R$ and nonzero elements $a, b \in R$ such that $0a = b$.
Solution: No such example exists, since it is a basic property of rings that $0a = 0$ for any $a$.

0b. A ring $R$ which is a commutative domain with identity, and an element $a \in R$ which is not a unit.
Solution: Let $R = \mathbb{Z}$, and take $a = 2$.

1. Nonzero elements $a, b$ in the ring $R = \mathbb{Z} \oplus \mathbb{Z}$ such that $ab = 0$.
Take $a = (1, 0)$, $b = (0, 1)$. Then $ab = (0, 0)$, which is the zero elt. in $\mathbb{Z} \oplus \mathbb{Z}$.

2. An element $x = a + b\sqrt{2}$ in the ring $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ such that $(x)(2 - \sqrt{2}) = 1$.

$$x = 1 + \frac{1}{2}\sqrt{2}.$$
3. A ring $R$ which has no identity element for multiplication.

$\mathbb{Z}$.

4. A commutative domain with identity $R$ which has characteristic 8.

There is no such ring, because the characteristic of a domain is either 0 or a prime $p$, and 8 is not prime.

5. An infinite ring $R$ and an ideal $I$ of $R$ such that the factor ring $R/I$ is finite.

Take $R=\mathbb{Z}$, $I=2\mathbb{Z}$. Then $R/I$ has only 2 elements.
6. An element \( f \) in the polynomial ring \( \mathbb{R}[x] \) such that \( (f)(x^2 + 5 + 1) = 1 \).

There is no such \( f \), since \( \mathbb{U}(\mathbb{R}[x]) = \mathbb{R}^* \), and so \( x^2 + 5 + 1 \) is not a unit in \( \mathbb{R}[x] \).

7. An element \( a \in \mathbb{Z} \) such that \( a \neq 1 \), and \( a \) is a unit.

The only choice is \( a = -1 \).

8. An ideal \( I \) of \( \mathbb{R}[x] \) such that \( (x - 5) \in I \), but \( (x - 5)(x^2 + 3) \notin I \).

This is impossible, for if \( (x - 5) \in I \), then \( f \cdot (x - 5) \in I \) for all \( f \in \mathbb{R}[x] \).
9. A field $F$ which has finitely many elements.

$\mathbb{Z}_p$.

10. A nonzero element $a \in \mathbb{Z}_{10}$ which is a zerodivisor.

$2$, since $2 \cdot 5 = 10 = 0 \text{ mod } \mathbb{Z}_{10}$. 

Part II: Long answers

11. (30 points) Let \( \mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\} \) be the ring of Gaussian integers, and let \( I \) be the principal ideal \( \langle 3+i \rangle \subset \mathbb{Z}[i] \). This problem is about the factor ring \( R = \mathbb{Z}[i]/I \). By the methods in the book or in class, one can show in fact that \( R \cong \mathbb{Z}_{10} \). This problem only asks you to do part of that argument.

   a. (10 points) For an integer \( a \in \mathbb{Z} \), think of \( a \) as the element \( a + 0i \in \mathbb{Z}[i] \). Show that \( 10 \in I \).

      Note that \( (3+i)(3-i) = 10 \), which is in \( I \) since it is a product of an element of \( I \) and a ring element.

   b. (10 points) Show that any coset \( (a+bi) + I \in R \) with \( a, b \in \mathbb{Z} \) is equal to a coset \( (c+0i) + I \in R \) for some \( c \in \mathbb{Z} \).

      Since \( 3+i \in I \), we have \( i + I = -3 + I \), whence

      \[
      (a+bi) + I = (a+i) + (b+I)(i+I)
      = (a+i) + (b+I)(-3+i)
      = (a-3b) + I, \text{ and } a-3b \in \mathbb{Z}.
      \]
c. (10 points) Using part (a) and (b), show that $R$ has finitely many elements. (Note that you do not need to prove that $R$ has exactly 10 elements.)

From (a) and (b), we know that every element of $R$ is equal to a coset of the form $c + I$, with $0 \leq c \leq 10$. This shows that $|R| \leq 10$. 
12. **(30 points)** Let $M_2(\mathbb{Z})$ be the ring of $2 \times 2$-matrices with integer entries. Let $R$ be the following subring of $M_2(\mathbb{Z})$:

$$R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

It turns out that $R$ is a *commutative* ring. You can assume that below, you don’t need to prove it (though it’s not hard.)

**a. (10 points)** Show that $\phi : R \to \mathbb{Z}$ defined by $\phi \left( \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = (a - b)$ is a homomorphism of rings.

Choose $x = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $y = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ in $R$ and compute:

$$\phi(x+y) = \phi \left( \begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix} \right) = a+c - (b+d)$$

$$= (a-b) + (c-d) = \phi(x) + \phi(y)$$

and also,

$$\phi(xy) = \phi \left( \begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix} \right) = ac+bd - (ad+bc)$$

$$= (a-b)(c-d) = \phi(x) \phi(y).$$

This shows that $\phi$ is a homomorphism.
b. (10 points) Prove that $R/(\ker \phi)$ is isomorphic to $\mathbb{Z}$.

Given $n \in \mathbb{Z}$, we have $n = \phi([n]\mathbb{Z})$, and so $\phi$ is onto $\mathbb{Z}$. We then have that $R/\ker \phi \cong \mathbb{Z}$ by the 1st isomorphism theorem.

c. (10 points) Is $\ker \phi$ a maximal ideal? Why or why not?

$\ker \phi$ is not maximal because $R/\ker \phi$ is not a field. (It is prime, however, since $\mathbb{Z}$ is a domain).