Part 1: Short answers (4 points each)

For each question, either find an example of a ring, element, etc. with the requested properties (no explanation necessary); or else if no such example exists, give a one sentence explanation of why it is impossible.

Sample problems:

0a. A ring R and nonzero elements $a, b \in R$ such that $0a = b$.

Solution: No such example exists, since it is a basic property of rings that $0a = 0$ for any a.

0b. A ring R which is a commutative domain with identity, and an element $a \in R$ which is not a unit.

Solution: Let $R = \mathbb{Z}$, and take $a = 2$.

1. Nonzero elements a, b in the ring $R = \mathbb{Z} \oplus \mathbb{Z}$ such that $ab = 0$.

Take $a = (1,0)$, $b = (0,1)$. Then $ab = (0,0)$, which is the zero elt. in $\mathbb{Z} \oplus \mathbb{Z}$.

2. An element $x = a + b\sqrt{2}$ in the ring $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ such that $(x)(2 - \sqrt{2}) = 1$.

$x = 1 + \frac{1}{2}\sqrt{2}$.

3. A ring R which has no identity element for multiplication.

$2\mathbb{Z}$.

4. A commutative domain with identity R which has characteristic 8.

There is no such ring, because the characteristic of a domain is either 0 or a prime p, and 8 is not prime.

5. An infinite ring R and an ideal I of R such that the factor ring R/I is finite.

Take $R = \mathbb{Z}$, $I = 2\mathbb{Z}$. Then R/I has only 2 elements.

6. An element f in the polynomial ring $\mathbb{R}[x]$ such that $(f)(x^2+5+1) = 1$.

There is no such f, since $U(\mathbb{R}[x]) = \mathbb{R}^*$, and so x^2+5+1 is not a unit in $\mathbb{R}[x]$.

7. An element $a \in \mathbb{Z}$ such that $a \neq 1$, and a is a unit.

The only choice is $a = -1$.

8. An ideal I of $\mathbb{R}[x]$ such that $(x-5) \in I$, but $(x-5)(x^2+3) \notin I$.

This is impossible, for if $(x-5) \in I$, then $f \cdot (x-5) \in I$ for all $f \in \mathbb{R}[x]$.

9. A field F which has finitely many elements.

$\mathbb{Z}_p$.

10. A nonzero element $a \in \mathbb{Z}_{10}$ which is a zerodivisor.

2, since $2 \cdot 5 = 10 = 0$ in $\mathbb{Z}_{10}$.

Part II: Long answers

11. **(30 points)** Let $\mathbb{Z}[i] = \{a+bi | a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers, and let I be the principal ideal $\langle 3 + i \rangle \subset \mathbb{Z}[i]$. This problem is about the factor ring $R = \mathbb{Z}[i]/I$. By the methods in the book or in class, one can show in fact that $R \cong \mathbb{Z}_{10}$. This problem only asks you to do part of that argument.

a.**(10 points)** For an integer $a \in \mathbb{Z}$, think of a as the element $a+0i \in \mathbb{Z}[i]$. Show that $10 \in I$.

Note that $(3+i)(3-i) = 10$, which is in I since it is a product of an element of I and a ring element.

b. **(10 points)** Show that any coset $(a + bi) + I \in R$ with $a, b \in \mathbb{Z}$ is equal to a coset $(c + 0i) + I \in R$ for some $c \in \mathbb{Z}$.

Since $3+i \in I$, we have $i+I = -3+I$, whence

$$\begin{aligned}(a+bi)+I &= (a+I) + (b+I)(i+I) \\ &= (a+I) + (b+I)\cdot(-3+I) \\ &= (a-3b) + I, \text{ and } a-3b \in \mathbb{Z}.\end{aligned}$$

c. **(10 points)** Using part (a) and (b), show that R has finitely many elements. (Note that you do not need to prove that R has exactly 10 elements.)

From (a) and (b), we know that every element of R is equal to a coset of the form $c+I$, with $0 \le c \le 10$. This shows that $|R| \le 10$.

12. **(30 points)** Let $M_2(\mathbb{Z})$ be the ring of 2×2-matrices with integer entries. Let R be the following subring of $M_2(\mathbb{Z})$:

$$R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$$

It turns out that R is a *commutative* ring. You can assume that below, you don't need to prove it (though it's not hard.)

a. **(10 points)** Show that $\phi : R \to \mathbb{Z}$ defined by $\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = (a - b)$ is a homomorphism of rings.

chose $x = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $y = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ in R and compute:

$$\phi(x+y) = \phi\left(\begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}\right) = a+c-(b+d)$$

$$= (a-b) + (c-d) = \phi(x) + \phi(y)$$

and also,

$$\phi(xy) = \phi\left(\begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix}\right) = ac+bd-(ad+bc)$$

$$= (a-b)(c-d) = \phi(x)\,\phi(y).$$

This shows that ϕ is a homomorphism.

b. **(10 points)** Prove that $R/(\ker\phi)$ is isomorphic to $\mathbb{Z}$.

Given $n \in \mathbb{Z}$, we have $n = \phi\left(\begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}\right)$, and so ϕ is onto $\mathbb{Z}$. We then have that $R/\ker\phi \hat{=} \mathbb{Z}$ by the 1^{st} isomorphism theorem.

c. **(10 points)** Is $\ker\phi$ a maximal ideal? Why or why not?

$\ker\phi$ is not maximal because $R/\ker\phi$ is not a field. (it is prime, however, since $\mathbb{Z}$ is a domain).