Part 1: Short answers (5 points each)

In these problems, no justification is necessary, just provide the answer. *No lengthy calculations should be necessary in any of the problems in this section.*

1. You are told that exactly one of the following rings is *not* a UFD. Which one is it?

$\mathbb{Z}[i]$, $\mathbb{Z}[x]$, $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}$.

$\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

2. A Euclidean domain is automatically also a (circle all that apply:)

PID, field, UFD.

3. You are told that $x^4 + x + 1$ is irreducible over $\mathbb{Z}_2$. How many elements does the factor ring $\mathbb{Z}_2[x]/\langle x^4 + x + 1\rangle$ have?

$2^4 = 16.$

4. The factor ring $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is isomorphic to what familiar field?

$\mathbb{C}$

5. Let $f = x^6 + 11$ and $g = x - 10$ in the ring $\mathbb{R}[x]$. Suppose one performs the division algorithm and finds $q, r \in \mathbb{R}[x]$ such that $f = qg + r$, where $\deg r < \deg g$ or $r = 0$. What is the remainder r?

$10^6 + 11$

Part II: Long answers

6. (25 points) Let F be a field. Prove that the polynomial ring $F[x]$ is a PID.

Since F is a field, we know that $F[x]$ is a domain, so it only remains to see that every ideal is principal. Let $I \trianglelefteq F[x]$. If $I = (0)$ then I is clearly principal, so suppose $I \neq (0)$. Among all elements of I, choose one, $g(x)$ say, of minimal degree. Then since $g \in I$, $\langle g \rangle \subseteq I$. For the reverse inclusion, let f be any element of I. By the division algorithm, we can find $q, r \in F[x]$ such that $f = q \cdot g + r$ and either $\deg r < \deg g$, or $r = 0$. Note that $r = f - q \cdot g \in I$ and so by our choice of g, $r = 0$. Thus $f \in \langle g \rangle$ and we conclude that $I = \langle g \rangle$.

7a. (20 points) Is $2x^3 + 6x^2 + 7x - 20$ an irreducible polynomial in $\mathbb{Q}[x]$? Justify your answer.

We use the mod p irreducibility test with $p=3$. The image of the polynomial in $\mathbb{Z}_3[x]$ is $2x^3+x+1$, which has no roots in $\mathbb{Z}_3$, hence is irreducible over $\mathbb{Z}_3$ since it has degree ≤ 3. Thus our original polynomial is irreducible over $\mathbb{Q}$.

7b. (5 points) Is the factor ring $\mathbb{Q}[x]/\langle 2x^3 + 6x^2 + 7x - 20\rangle$ a field? Justify your answer.

The polynomial is irreducible by (a), so the ideal it generates is maximal, and thus the corresponding quotient ring is a field.

In the following problem, we work with the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$. This ring can also be thought of as $\mathbb{Z}[\sqrt{-1}]$. It has a norm function $N(a + bi) = a^2 + b^2$. You may quote and use without proof the basic properties of the norm function.

8. **(25 points)** Write the element 6 as a product of irreducibles in $\mathbb{Z}[i]$. Justify your answer (i.e. prove the elements in your factorization really are irreducible.)

$6 = 3 \cdot 2$, but 2 factors as $(1+i)(1-i)$ in $\mathbb{Z}[i]$, so it remains to see that 3, $(1+i)$, and $(1-i)$ are irreducible in $\mathbb{Z}[i]$. $(1 \pm i)$ is irreducible because it has norm 2, which is prime. If 3 were reducible, then we would have $3 = x \cdot y$ with $n(x) = n(y) = 3$, however, the equation $a^2 + b^2 = 3$ has no solutions in $\mathbb{Z}$, and so 3 is irreducible.