Math 109 Winter 2007 Exam 1

January 31, 2007

NAME: ANSWERS

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Problem 1 (25 pts)

In this problem, let u_n be the nth Fibonacci number. These numbers are defined inductively by setting $u_1 = 1$, $u_2 = 1$, and $u_{n+1} = u_n + u_{n-1}$ for $n \geq 2$.

Prove that the inequality $u_n < 3^n$ holds for all $n \geq 1$. Be absolutely clear about what method you are using and make the various steps of the method clear in your proof.

We prove the result by strong induction. When $n = 1$, then $u_1 = 1 < 3^1 = 3$. When $n = 2$, then $u_2 = 1 < 3^2 = 9$. This proves the base cases.

Now assume that $u_m < 3^m$ holds for all $1 \leq m \leq k$, for some $k \geq 2$. Applying this for $m = k$ and $m = k - 1$, we get that both $u_k < 3^k$ and $u^{k-1} < 3^{k-1}$ are true. Now since $u_{k+1} = u_k + u_{k-1}$, we have

$$u_{k+1} = u_k + u_{k-1} < 3^k + 3^{k-1} < 3^k + 3^k < 3^k + 3^k + 3^k = 3(3^k) = 3^{k+1}.$$

This completes the proof of the induction step.

Thus $u_n < 3^n$ holds for all integers $n \geq 1$.

Problem 2 (25 pts)

Let P and Q be arbitrary propositions.

(a)(5 pts). Consider the implication $P \Rightarrow Q$. Write down the *converse* of $P \Rightarrow Q$ and the *contrapositive* of $P \Rightarrow Q$, clearly labeling each.

The converse of $P \Rightarrow Q$ is the proposition $Q \Rightarrow P$.

The contrapositive of $P \Rightarrow Q$ is the proposition $(\text{not}\, Q) \Rightarrow (\text{not}\, P)$.

(b)(10 pts). Write down a full truth table for each of the three propositions in part (a) above ($P \Rightarrow Q$, its converse, and its contrapositive.)

We put all of the needed information in a single truth table. (The columns not P and not Q were not required but were included to make it easier to calculate the column $(\text{not}\, Q) \Rightarrow (\text{not}\, P)$.)

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\text{not}\, Q)$	$(\text{not}\, P)$	$(\text{not}\, Q) \Rightarrow (\text{not}\, P)$
T	T	T	T	F	F	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

(c)(10 pts). Suppose that $P \Rightarrow Q$ is known to be *false*. Is the converse true, false, or can't you tell? Is the contrapositive true, false, or can't you tell? Write a *short* justification for your answers which refers to the truth tables in part (b).

We read the answer directly off the truth table in the previous part. The only row of the table in which $P \Rightarrow Q$ is false is the second row. In that row, the converse $Q \Rightarrow P$ is true, and the contrapositive $(\text{not}\, Q) \Rightarrow (\text{not}\, P)$ is false.

Problem 3 (25 pts)

Let A, B, and C be any three sets. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Prove this directly *in words* from the basic definitions of set theory. (You are not allowed to quote theorems from the book, since this statement is itself a theorem in the book.)

First we prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Suppose that $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in B \cap C$. Suppose that $x \in A$; then certainly both $x \in A \cup B$ and $x \in A \cup C$. Thus $x \in (A \cup B) \cap (A \cup C)$. The other case is where $x \in B \cap C$. In this case $x \in B$ and $x \in C$. Since $x \in B$, we have $x \in A \cup B$, and since $x \in C$, we have $x \in A \cup C$. So $x \in (A \cup B) \cap (A \cup C)$ in this case also. Altogether, we have proven that $x \in A \cup (B \cap C)$ implies $x \in (A \cup B) \cap (A \cup C)$ and so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ as claimed.

Now we prove that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Suppose that $x \in (A \cup B) \cap (A \cup C)$. Then both $x \in A \cup B$ and $x \in A \cup C$. Suppose first that $x \in A$. Then $x \in A \cup (B \cap C)$ as we want. So the remaining case is when $x \notin A$. But since $x \in A \cup B$, this forces $x \in B$, and since $x \in A \cup C$, this forces $x \in C$. Thus $x \in B \cap C$ and so $x \in A \cup (B \cap C)$ in this case also. Thus we see that $x \in (A \cup B) \cap (A \cup C)$ implies $x \in A \cup (B \cap C)$ and so $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Since we have proven each set is a subset of the other, it follows that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ as required.

Problem 4 (25 pts)

(a)(15 pts) Prove that for all integers a and b, if a is even and $a|b$, then b is even. Prove this directly from the definitions (do not quote theorems or homework exercises.)

Let a and b be integers where a is even and $a|b$. By definition, a is even means there is some $q \in \mathbb{Z}$ with $a = 2q$. Also by definition, $a|b$ means there is some $p \in \mathbb{Z}$ with $b = pa$. Then $b = 2pq = 2(pq)$, where pq is also an integer. So b is also even by definition.

(b)(10 pts) Prove that for all integers c and d, if d is odd and $c|d$, then c is odd. Again do not quote theorems from the book or homework exercises, but feel free to quote the result of part (a).

We prove this by contradiction. Let c and d be integers where d is odd and $c|d$. Assume that c is *not* odd, and so c is even. Then c is even and $c|d$, which by the result of part (a) implies that d is even. But we know that d is odd and so this is a contradiction. Thus our assumption that c is not odd was false, and so c must be odd.