Math 109 Spring 06 Exam 1

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Problem 1 (30 points)

Prove that $\sqrt{2}$ is an irrational number.

Solution: *In this problem and the next, I accept without proof that every integer is either even or odd. (This can be proved using the division algorithm.) In problem 1, one also needs to know the fact that if n is an integer and n^2 is even, then n is even. I proved this in the solution below: but since this is the contrapositive to a fact we proved in class, I allowed this fact to be stated without proof as well without loss of credit. If you proved this fact anyway, great.*

The proof of the result is by contradiction. Assume that $\sqrt{2}$ is a rational number. Then $\sqrt{2} = a/b$ for some integers a and b with $b \neq 0$. In addition, we can assume that the fraction a/b is written in lowest terms so that a and b have no common factors greater than 1.

Squaring both sides of $\sqrt{2} = a/b$, we obtain $2 = a^2/b^2$, and so $2b^2 = a^2$. Thus a^2 is even. We claim that a is also even. Suppose not; then a is odd, and so $a = 2m + 1$ for some integer m, in which case $a^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ is also odd, a contradiction. So a is even as claimed.

Thus we can write $a = 2k$ for some integer k. Substituting back in to our equation, we have $2b^2 = 4k^2$. Dividing by 2, $b^2 = 2k^2$. So b^2 is even, and by the same argument as in the last paragraph, b is even. We have proved that both a and b are even, so the fraction a/b is not in lowest terms, which is a contradiction.

Thus our assumption that $\sqrt{2}$ is rational must have been false, and we conclude that $\sqrt{2}$ is irrational.

Problem 2 (25 points)

Prove the following statement: For every $n \in \mathbb{N}$, n is odd if and only if $4|(n^2-1)$.

Solution: Let n be odd. Then by definition, $n = 2k+1$ for some integer k. Squaring, $n^2 = 4k^2+4k+1$, and so $n^2-1 = 4k^2+4k = 4(k^2+k)$. This is clearly a multiple of 4, so $4|(n^2-1)$ as required.

Conversely, let $4|(n^2-1)$ for some natural number n. Suppose that n is even. Then $n = 2k$ for some integer k, and then $n^2 = 4k^2$. So $4|n^2$. The difference of two multiples of 4 is again a multiple of 4, so we conclude that $4|(n^2)-(n^2-1)$, in other words $4|1$. This is absurd, so we have a contradiction. Thus our supposition that n is even was incorrect, and n must be odd.

Problem 3 (25 points)

3a (15 points). Are the propositional expressions

$$(\neg P) \to (Q \vee P) \quad \text{and} \quad \neg(P \wedge Q)$$

logically equivalent? If so, prove it. If not, give a counterexample.

Solution: The two expressions are not logically equivalent. One way is to prove this using a truth table, but we choose to reason in English.

Suppose that P is true and Q is true. Then $\neg P$ is false, and since a conditional statement is automatically true when the hypothesis is false, the statement $(\neg P) \to (Q \vee P)$ is true.

On the other hand, when P and Q are both true, then $P \wedge Q$ is also true, and so $\neg(P \wedge Q)$ is false.

Since the two expressions do not take on the same truth values for all possible choices of truth values of the variables, they are not logically equivalent expressions.

3b (10 points). If the following proposition is true, prove it. If it is false, give a counterexample.

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } xy \neq 0.$$

Solution: The proposition is false. If we choose $x = 0$, then no matter what y is, the product $xy = 0$. Thus for the choice $x = 0$, there does not exist $y \in \mathbb{Z}$ with the property $xy \neq 0$.

Problem 4 (20 points)

For each $n \in \mathbb{N}$, let A_n be the set $\{x \in \mathbb{Z} \big| \; n \text{ divides } x\}$. (Suggestion: List explicitly the first few of the sets $A_1, A_2, A_3, A_4, \ldots$, so that you can better visualize what these sets look like.)

4a (10 points). Calculate $\displaystyle\bigcap_{n \in \mathbb{N}} A_n$ and prove your answer.

Solution: To help us visualize things, we first calculate

$$\begin{aligned} A_1 &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \\ A_2 &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}, \\ A_3 &= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} \end{aligned}$$

after which we believe we understand the pattern.

We claim that $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$.

First, note that $0 \in A_n$ for all $n \in \mathbb{N}$, because $n|0$ holds for all natural numbers. So $0 \in \bigcap_{n \in \mathbb{N}} A_n$.

On the other hand, suppose $x \in \bigcap_{n \in \mathbb{N}} A_n$. Then x must be a multiple of every natural number n. But choosing any natural number $n > |x|$, it is impossible for x to be a nonzero multiple of that n. Thus the only possibility is $x = 0$, and thus 0 is the only element in $\bigcap_{n \in \mathbb{N}} A_n$.

In conclusion, we have $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ as claimed.

4b (10 points). Calculate $\displaystyle\bigcup_{n \in \mathbb{N}} A_n$ and prove your answer.

Solution: We claim that $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{Z}$. Since each A_n consists of integers, clearly $\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{Z}$. On the other hand, given any integer x, 1 divides x, and so $x \in A_1$. This proves that $\mathbb{Z} \subseteq A_1$, and since A_1 is one of the sets in the union, this forces $\mathbb{Z} \subseteq \bigcup_{n \in \mathbb{N}} A_n$. Altogether, we must have $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{Z}$ as claimed.