

Math 109 Spring 06 Exam 2

May 24, 2006

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Problem 1 (25 points)

In this problem, your final answer can involve standard combinatorial symbols, you don't have to evaluate these to get a number. In the game show "500-ball or no 500-ball", there are three bins with circulating numbered balls. There are 500 balls in each bin, numbered from 1 to 500; the balls in the first bin are green, those in the second bin are blue, and those in the last are red.

(a)(10 points). Suppose 3 balls are chosen randomly from the green bin. How many possible choices are there which contain the ball numbered 500? (The three balls are chosen all at once, and two choices are the same if they contain the same set of three balls.) Explain your answer.

Solution. The choice of 3 balls must contain the 500-ball, but the other two balls can be anything from among the remaining 499 balls. Therefore the total number of possibilities is the total number of ways of choosing 2 out of 499 balls, which is $\binom{499}{2}$.

(b)(10 points). To play the game, a contestant pulls a lever which releases three random balls from each bin, so nine balls total. The contestant wins if at least one ball (of any color) numbered 500 shows up in the drawing. How many different *losing* combinations are there? (Again, two combinations are the same if they contain the same 9 balls.) Explain your calculations.

Solution. One loses if no balls numbered 500 show up. The number of choices of 3 balls from the green hopper without the 500-ball is equal to the number of ways of choosing 3 balls from the other 499 balls, namely $\binom{499}{3}$. (Alternatively, we could count the total number of choices of 3 random balls, which is $\binom{500}{3}$, and subtract the number of choices containing the 500-ball, which is $\binom{499}{2}$ by part (a). Then the number of choices of 3 balls not containing the 500-ball is $\binom{500}{3} - \binom{499}{2}$. This is also equal to $\binom{499}{3}$ by the rule for Pascal's triangle.)

Since the same argument works for each hopper, the product rule gives that the total number of losing combinations is $\binom{499}{3} \binom{499}{3} \binom{499}{3}$.

(c)(5 points). In the game described in part (b), how many winning combinations are there? Explain your calculations.

Solution. We could use inclusion/exclusion, but there is no reason to do so much work. The total number of possible combinations is $\binom{500}{3} \binom{500}{3} \binom{500}{3}$ by the product rule, and $\binom{499}{3} \binom{499}{3} \binom{499}{3}$ of them are losing, by part (b). So the total number of winning combinations is $\binom{500}{3} \binom{500}{3} \binom{500}{3} - \binom{499}{3} \binom{499}{3} \binom{499}{3}$.

Problem 2 (25 points)

Recall that the fibonacci sequence is defined by declaring $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$. Prove that given a natural number n , the n th fibonacci number f_n is even if and only if n is a multiple of 3.

Solution. We prove this by induction on n . If $n = 1$ or $n = 2$, then $f_n = 1$, which is odd, and n is not a multiple of 3 in either case. So the theorem is true when $n = 1$ or $n = 2$. This shows the base cases of the induction.

Now for each $n \geq 3$, assuming the theorem is true for all k with $1 \leq k \leq n$, we show it is true for $n + 1$. Then the theorem will be true for all $n \in \mathbb{N}$ by the principle of strong induction.

So assume that the theorem is true for all k with $k \leq n$. Consider f_{n+1} . If $n + 1$ is a multiple of 3, then n and $n - 1$ are not multiples of three, so the induction hypothesis gives that f_n and f_{n-1} are both odd. Then $f_{n+1} = f_n + f_{n-1}$ is even. On the other hand, if $n + 1$ is not a multiple of 3, then exactly one of the numbers n and $n - 1$ is a multiple of three. So by the induction hypothesis, one of the numbers f_n, f_{n-1} is even and the other one is odd. So $f_n + f_{n-1}$ is odd. In either case, we see that f_{n+1} satisfies the theorem, so this completes the proof of the induction step.

Problem 3 (25 points)

Let S be a set with a relation R . Recall what this means: R is some rule which assigns true or false to the expression xRy for each ordered pair of elements $x, y \in S$.

(a)(10 points). Define what it means for R to be an equivalence relation.

Solution. R is an equivalence relation if the following three axioms hold:

(Reflexivity) For every $x \in S$, xRx is true.

(Symmetry) If xRy is true for some $x, y \in S$, then yRx is also true.

(Transitivity) If xRy and yRz are true for some $x, y, z \in S$, then xRz is true.

(b)(15 points). Suppose that R is an equivalence relation, so we write \sim instead of R . Let x, y be elements of S such that $x \sim y$. Using only the axioms for an equivalence relation, prove that the equivalence classes $[x]$ and $[y]$ are equal.

Solution. Recall that $[x] = \{z \in S \mid z \sim x\}$. Now suppose that $z \in [x]$. Then $z \sim x$. Since also $x \sim y$, $z \sim y$ by transitivity. Thus $z \in [y]$. This shows that $[x] \subseteq [y]$.

On the other hand, suppose that $z \in [y]$. Then $z \sim y$. By symmetry, since $x \sim y$ is known, $y \sim x$ is also true. Then by transitivity, $z \sim x$, so $z \in [x]$. This shows that $[y] \subseteq [x]$.

Since we have shown both inclusions of sets, we must have $[x] = [y]$.

Problem 4 (25 points)

Annoyed by all the spam he gets from Washington Mutual bank, a hacker infiltrates the bank's computer system with a virus. The virus makes it so that the bank's computer system will only allow deposits which are whole-dollar multiples of \$14, and withdrawals which are whole-dollar multiples of \$59.

(a)(10 points). Using the Euclidean Algorithm, compute $\gcd(14, 59)$. (Use the Euclidean algorithm even if another method would give the answer more quickly.)

Solution

$$59 = 4(14) + 3$$

$$14 = 4(3) + 2$$

$$3 = 1(2) + 1$$

$$2 = 2(1) + 0.$$

Then $\gcd(14, 59) = 1$ is the last nonzero remainder in the calculation.

(b)(15 points). Show how a customer, through a clever combination of deposits and withdrawals, can still manage to make a net deposit of \$100. Show your work and briefly explain the methods you use.

Solution

We want to find a positive x and a negative y so that $14x + 59y = 100$.

First, we find any integers $a, b \in \mathbb{Z}$ such that $14a + 59b = 1$. This is done by reversing the Euclidean algorithm:

$$1 = 1(3) - 1(2)$$

$$= 1(3) - 1(14 - 4(3)) = 5(3) - 1(14)$$

$$= 5(59 - 4(14)) - 1(14) = -21(14) + 5(59).$$

However, we want a negative multiple of 59 and a positive multiple of 14. By a theorem we learned in class, we know that all integer solutions to the equation $14a + 59b = 1$ are given by $a = -21 + 59t, b = 5 - 14t$, as t ranges over all integers. Taking $t = 1$, we get that $a = 38, b = -9$ is another solution. So $38(14) - 9(59) = 1$. Then $3800(14) - 900(59) = 100$.

So we deposit $3800(14)$, and withdraw $900(59)$, and that does the trick. (With more work, you can find other solutions involving smaller numbers.)