Math 109 Spring 2006 HW 7

HW Due **Friday** 5/26/06 in class

Exercise and page numbers refer to Fletcher and Patty, 3rd edition. Check them against a friend's book if you have another edition. Working with a classmate is fine, but the final writeup should be your own work.

Part 0

These are suggested exercises for extra practice if you feel you need it. Do not hand these exercises in.

FP, Chapter 4, #67, 68, 72, 82, 84
FP, Chapter 5, #1, 7

Part 1

Turn these exercises in.
FP, Chapter 4, #80, 81, 83
FP, Chapter 5, #6, 9, 17

Part 2

These are more challenging and/or longer exercises.

1. Let p be an odd prime number. We say that $a \in \mathbb{Z}$ is a sqaure modulo p if there exists some $b \in \mathbb{Z}$ such that $b^2 \equiv a \mod p$. Prove that among the numbers $\{0, 1, 2, \ldots, p-1\}$, exactly $(p+1)/2$ are squares modulo p.

2. The "First year calculus student's dream" is the tempting formula $(x + y)^n = x^n + y^n$, which is, of course, false in general (otherwise we would not need the binomial theorem.) However, there is a variant of the calculus student's dream which *is* correct. Prove that if p is a prime number and $a, b \in \mathbb{Z}$, then $(a + b)^p \equiv a^p + b^p \mod p$.

3. In this problem, you will explore injective and surjective are related for functions from a finite set to itself (these observations are useful to remember.)

(a). Prove that if S is a set with finitely many elements and $f : S \to S$ is an injective function, then f is actually bijective. Give an example which shows that the same result does not hold in general for infinite sets S.

(b). Prove that if S is a set with finitely many elements and $f : S \to S$ is an surjective function, then f is actually bijective. Give an example which shows that this result does not hold in general for infinite sets S.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is called *increasing* if $f(a) \leq f(b)$ whenever $a < b$, and *strictly increasing* if $f(a) < f(b)$ whenever $a < b$.

(a). Show by example that an increasing function $f : \mathbb{R} \to \mathbb{R}$ is not necessarily injective.

(b). Show that a strictly increasing function f is injective. Give an example of a strictly increasing function which is not surjective.

(c). By part (b), if f is a strictly increasing function which is also surjective, then f is bijective. In this case, show that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is also strictly increasing.

5. Let A, B, C be sets. Let $f : A \to B$, $g : B \to C$, and $h : A \to B$ be functions.

(a). Prove that if $g \circ f$ is injective, then f is injective. Show by example that $g \circ f$ being injective does not force g to be injective.

(b). Prove that if f and g are both injective, then $g \circ f$ is injective.

(c). Prove that if g is injective and $g \circ f = g \circ h$, then $f = h$.