Math 109 Spring 2006 HW 8

HW Due Wednesday 6/7/06 in class

Exercise and page numbers refer to Fletcher and Patty, 3rd edition. Check them against a friend's book if you have another edition. Working with a classmate is fine, but the final writeup should be your own work.

Part 1

Turn these exercises in.

FP, Chapter 5, #68, 74, 77

FP, Chapter 7, #38, 40

Part 2

These are more challenging and/or longer exercises.

1. Let $f : A \to B$ be a function. Let $C_1, C_2 \subseteq A$ be any two subsets of A, and $D_1, D_2 \subseteq B$ be any two subsets of B. In this problem you will prove selected parts of the theorem stated in class about how image and inverse image interact with intersection and union. (This is also Theorem 5.12 in the book, which states things for arbitrary collections of sets instead of just two sets.)

(a). Prove that $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(b). Prove that $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

(c). If f is one-to-one, then prove $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

2. In this problem, you will prove that all open intervals in the set of real numbers have the same cardinality as the set of all real numbers. Recall that if $a, b \in \mathbb{R}$ and $a < b$, then we define the *open interval*

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

(a). Let a, b, c, d be four real numbers with $a < b$, and $c < d$. Prove that the sets (a, b) and (c, d) have the same cardinality by finding a one-to-one and onto function $f : (a, b) \to (c, d)$.

(b). Show that $\mathbb{R}$ has the same cardinality as the open interval $(-\pi/2, \pi/2)$, by finding an explicit one-to-one and onto function $f : (-\pi/2, \pi/2) \to \mathbb{R}$. (You should explain why you know that the f you choose is one-to-one and onto, but you do not have to prove it rigorously.)

(c). Let a, b be any two real numbers with $a < b$. Prove that the interval (a, b) and $\mathbb{R}$ have the same cardinality.

3. This problem considers certain collections of subsets of $\mathbb{N}$.

(a). Let S be the collection of all *finite* subsets of $\mathbb{N}$. Prove that S is countable.

(b). Let P be the collection of all subsets of $\mathbb{N}$ (in other words, P is the power set $\mathcal{P}(\mathbb{N})$.) Prove that P is uncountable.

(c). Let T be the collection of all *infinite* subsets of $\mathbb{N}$. Prove that T is uncountable.

4. A *polynomial function* is a function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_i \in \mathbb{R}$ for all i. The *degree* of f is the highest power of x appearing (so for the f above, the degree is n as long as $a_n \neq 0$.) The polynomial function f has *rational coefficients* if $a_i \in \mathbb{Q}$ for all i. A *root* of f is a value $b \in \mathbb{R}$ for which $f(b) = 0$.

The factor theorem in algebra says that a polynomial of degree n has at most n roots; assume this. A number $b \in \mathbb{R}$ is called *algebraic* if it is a root of some polynomial function with *rational* coefficients; otherwise b is called *transcendental*. For example, $\sqrt{2}/2$ is algebraic since it is a root of the degree 2 polynomial function $f(x) = x^2 - (1/2)$.

(a). Prove that the set of all algebraic real numbers is countable.

(b). Prove that there exists a transcendental real number.