The property “is a normal subgroup of” is not transitive in general; that is, if one has $H \trianglelefteq K$ and $K \trianglelefteq G$, it does not follow in general that $H \trianglelefteq G$. Recall that an automorphism of a group $G$ is an isomorphism from $G$ to itself. We say that a subgroup $H \leq G$ is characteristic in $G$, and write $H \char G$, if for all automorphisms $\phi$ of $G$, $\phi(H) = H$.

1. (a). Show that if $H \char G$, then $H \leq G$.
(b). Let $H < K < G$, where $H \char K$ and $K \leq G$. Then $H \leq G$.
(c). Show that if $K$ is a cyclic subgroup of $G$ and $K \leq G$, then every subgroup $H$ of $K$ satisfies $H \leq G$.

2.* Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the formulas $f(x) = -x$ and $g(x) = x + 1$. Let $G = \langle f, g \rangle$ be the subgroup of the group $S_\mathbb{R}$ of all permutations of the set $\mathbb{R}$ which is generated by $f$ and $g$. Prove carefully that

$$G \cong \langle a, b \mid b^2 = e, ba = a^{-1}b \rangle.$$  
(This group is called the infinite dihedral group, $A_\infty$.)