Math 200a Final, 12/9/2008

You do not have to do all problems to get a good score. I prefer complete and correct solutions to fewer problems than sketchy solutions to more problems. However, don't aim for perfection in your write-ups at the expense of not doing problems which you are able to do. You may quote results proved in class or in the textbook, but try to avoid quoting results proved only in homework exercises.

1. In this problem, you will study (part of) the classification of groups of order 12 in terms of semidirect products (Hungerford classified groups of order 12 by a more ad-hoc method.)

1a. Suppose that $|G| = 12$ and G does not have a normal Sylow 3-subgroup. Show that $G \cong A_4$, the alternating group on four symbols. (Hint: study an action of G on something.)

1b. Suppose that $|G| = 12$ and G has a normal Sylow 3-subgroup. Show that G is isomorphic to a semidirect product $\mathbb{Z}_3 \rtimes_\phi H$ for some group H of order 4 and homomorphism $\phi : H \to \mathrm{Aut}(\mathbb{Z}_3)$. Describe all of the possibilities for H and ϕ.

1c. Prove carefully that one of the semidirect products you found in part (b) is isomorphic to $\langle a, b | a^4 = e, b^3 = e, ab = b^{-1}a \rangle$.

2. A group G is called *polycyclic* if it has a subnormal series

$$\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

(so G_i is normal in G_{i+1}, but not necessarily normal in the whole group G), such that G_{i+1}/G_i is a cyclic group (finite or infinite) for all $0 \leq i < n$.

2a. Show that subgroups and factor groups of polycyclic groups are again polycyclic.

2b. Show that a finite group G is polycyclic if and only if it is solvable.

2c. Give an example, with proof, of an infinite group G which is solvable but not polycyclic.

3. Find, with proof, the smallest *odd* positive integer n such that there exists a group G of order n which is *not* nilpotent.

4a. Let $\mathcal{C}$ be a concrete category. Suppose A is an object of the category and $X \subseteq A$ is a subset. Define what it means for A to be *free* on the subset X.

4b. Let $\mathcal{C}$ be the category whose objects are all rings (with identity); whose morphisms are all ring homomorphisms which are unital, i.e. send the identity to the identity; and where composition of morphisms is just the usual composition of functions. You do not need to prove that $\mathcal{C}$ is a category. Show that the ring $\mathbb{Z}[x]$ is a free object in this category on the subset $\{x\}$.

5a. Let M be a left module over any ring R. Recall that for $m \in M$, the annihilator of m is $\mathrm{ann}_R(m) = \{r \in R | rm = 0\}$, and that M is a *torsion module* if $\mathrm{ann}_R(m) \neq (0)$ for all m in M. Now we define the *annihilator of* M to be $\mathrm{ann}_R(M) = \{r \in R | rm = 0 \text{ for all } m \in M\}$. (In other words, this is the set of ring elements that kill *all* elements of M.) Show that $\mathrm{ann}_R(M)$ is an ideal (i.e. both left and right ideal) of R.

5b. Suppose that M is a finitely generated left module over a PID R. State what the fundamental structure theorem for such modules (invariant factor form) tells you about M. Show that $\mathrm{ann}_R(M) \neq (0)$ if and only if M is a torsion module. In case $\mathrm{ann}_R(M) \neq (0)$, find a formula for $\mathrm{ann}_R(M)$ in terms of the invariant factors of M.

5c. Find an example, with proof, of a PID R and a left module M over R such that M is *not* finitely generated, with the property that $\mathrm{ann}_R(M) = (0)$ but M is nonetheless a torsion module.

6. Let $\mathrm{Mat}_2(\mathbb{C})$ be the ring of 2×2 matrices over the complex numbers. Let $\mathcal{S}$ be the set of all $A \in \mathrm{Mat}_2(\mathbb{C})$ which satisfy the additional properties that (i) A is invertible (i.e. $\det A \neq 0$); and (ii) A^2 is *similar* to A.

Show that $\mathcal{S}$ is equal to the union of exactly three distinct similarity classes of matrices. (Hint: think about the Jordan canonical form.)