

Math 200a Fall 2011 Homework 8

Due Friday 12/02/2011 by 5pm in homework box on 6th floor of AP&M

Upcoming schedule: We will cover factorization in the Gaussian integers, which is covered in the text in Section 8.3, on Monday 11/28. Additional HW problem 2 is based on that material. On Wednesday 11/30 we will talk about criteria for irreducibility of polynomials, in particular the Eisenstein criterion.

The final exam will be on Wednesday December 7, 3pm-6pm. It will be in our usual classroom unless I make an announcement otherwise.

Reading assignment: Finish reading Chapter 9, and anything earlier you didn't read or want to reread. We will cover only the very beginning of Section 9.6, namely Hilbert's basis theorem.

Exercises to be handed in: (all exercise numbers refer to Dummit and Foote, 3rd edition.)

Section 8.3 : **5, 6(c)** (Look over 6(a,b) too, but don't write them up.)

Section 9.3: **1, 2**

Section 9.4: **9, 11** (These problems use the Eisenstein criterion. We will cover this in class on Wednesday November 30. You can easily read the statement of the criterion in section 9.3 and work on these problems before that. Exercise #2 in this section is a good warmup. For #11, I suggest thinking of $\mathbb{Q}[x, y]$ as $(\mathbb{Q}[x])[y]$.)

Exercises not from the text: (to be handed in):

1. (a). let F be a field. Show that the polynomial ring in infinitely many variables over F , $R = F[x_1, x_2, x_3, \dots]$, is not a noetherian ring. (Recall that elements of this ring are finite linear combinations with F -coefficients of *monomials* of the form $x_{i_1}^{e_1} x_{i_2}^{e_2} \dots x_{i_m}^{e_m}$, for any $1 \leq i_1 < i_2 < \dots < i_m$ and any $e_m \geq 1$.)

(b). Show that if S is a noetherian ring, then so is S/I , for any ideal I . Thus the noetherian property passes to factor rings.

(c). Given an example of a ring S and a subring $R \subseteq S$ such that S is noetherian, but R is not. Thus the noetherian property does not pass to subrings. (Hint: take an example of a non-noetherian ring and embed it in a ring that is noetherian.)

2. Show that the equation $x^2 + 2y^2 = p$ has a solution in integers, where p is an odd prime, if and only if $\overline{-2}$ is a square of an element in the multiplicative group \mathbb{Z}_p^\times . (Hint: If $x^2 + 2y^2 = p$, show directly that $\overline{-2}$ is a square in \mathbb{Z}_p^\times through manipulations in this group. Conversely, if $\overline{-2}$ is a square in \mathbb{Z}_p^\times , quote that this group is cyclic and use similar ideas as we used in our analysis of the Gaussian integers to show that p is not prime in $R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$, and that a factorization of p in R leads to a solution to $x^2 + 2y^2 = p$. Note that you showed that R is a Euclidean domain in a previous exercise, so you can freely use that R is a UFD here.)

(BTW, it is known that -2 is a square mod p if and only if p is congruent to either 1 or 3 modulo 8.)