Math 200a Fall 2012 Final Exam

December 14, 2012

The exam may be too long to finish. Do the best as you can on as many problems as you can. You may use major theorems proved in class or the textbook, unless the point of the problem is to repeat the proof of such a result. Avoid quoting results proved only in homework exercises, when possible.

NAME:

Problem 1/ 10	
Problem 2 /15	
Problem 3 /10	
Problem 4 /15	
Problem 5 /10	
Problem 6 /10	
Total /70	

1. (10 pts) Let G be a finite group, and H a subgroup of G. Suppose that p is a prime dividing $|G|$, and let P be a Sylow p-subgroup of G.

(1a) (4 pts) Suppose P is normal in G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

(1b) (4 pts) Suppose instead that H is normal in G (but P now might not be normal). Prove that $P \cap H$ is a Sylow p-subgroup of H.

(1c) (2 pts) If neither P nor H is normal in G, must $P \cap H$ be a Sylow p-subgroup of H? Prove or give a counterexample.

2. (15 pts)

Let G be a *non-Abelian* group with $|G| = 8$ in this problem. This problem classifies all such groups.

(2a) (2 pts) Show that G has an element a of order 4.

(2b) (6 pts) If G contains an element b of order 2 such that $b \notin \langle a \rangle$, where a is the order 4 element of part (a), then prove that $G \cong D_8$ is isomorphic to the dihedral group with 8 elements.

(2c) (7 pts) If G has no element b as in part (b), then prove $G \cong Q_8$ is the quaternion group in this case. You may use without proof that $Q_8 \cong \langle x, y | x^2 = y^2, x^{-1}yx = y^{-1} \rangle$ is a presentation of Q_8.

3. (10 pts)

(3a) (5 pts) For which $n \geq 2$ is S_n a solvable group? For which $n \geq 2$ is S_n a nilpotent group? Justify your answers.

(3b) (5 pts) Prove that every group G with $|G| = 36$ is solvable.

4. (15 pts) Recall that if D is a nonzero squarefree integer, then the ring $\mathbb{Z}[\sqrt{D}] = \{a+b\sqrt{D} | a, b \in \mathbb{Z}\}$ always has a norm function $N(a+b\sqrt{D}) = |a^2 - Db^2|$ with special properties. You may use basic facts about this norm function without proof to streamline your proofs of the following, but say what you are using.

(4a) (8 pts) Prove that $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the norm function $N(a + b\sqrt{2}) = |a^2 - 2b^2|$.

(4b) (7 pts) Prove that the ring $\mathbb{Z}[\sqrt{-10}] = \{a + b\sqrt{-10} | a, b \in \mathbb{Z}\}$ is not a UFD.

5. (10 pts)

(5a) (5 pts) Let F be a field. Show that the formal power series ring $F[[x]] = \{\sum_{i=0}^{\infty} a_i x^i | a_i \in F\}$ is a local ring.

(5b) (5 pts) Show that the ring $F[[x]]$ is a PID.

6. (10 pts)

Let R be a commutative ring (with 1). Let Σ be the collection of all infinitely generated ideals I of R.

(6a) (5 pts) Show that if Σ is nonempty, then Σ contains a maximal element P under inclusion.

(6b) (5 pts) Show that P as constructed in (a) must be prime. Conclude that if R is a ring such that every prime ideal of R is finitely generated, then R is noetherian.

(Hint: to show that P is prime, suppose that $xy \in P$ with $x \notin P, y \notin P$. Show there is a finitely generated ideal $I \subseteq P$ such that $P + xR = I + xR$, and then that $P = I + xJ$ where $J = \{z \in R | zx \in P\}$. Acheive a contradiction.)