Math 200a Fall 2012 Homework 7

Due Friday 11/30/2012 by 12pm in homework box

Reading assignment: Chapter 8. We covered localization more generally than in chapter 7 in the text; note that the theory of localization in the generality we defined it is covered in the first few pages of section 15.4 in the text. We covered the definition of noetherian ring and equivalent characterizations of the noetherian property on November 21. This proof can be found in the text at Section 12.1, Theorem 1 (replace the module M by the ring R, and submodules of M by ideals of R.)

We will begin to cover the material on Euclidean domains, which is the subject of the exercises from Section 8.1 below, on Monday the 26th.

Exercises to be handed in: (all exercise numbers refer to Dummit and Foote, 3rd edition.)

Section 7.5: **5** (This exercise refers to the construction of the ring of Laurent series $F((x))$ which is given in exercise 5 of section 7.2. Since you did not have to do that exercise before, read it over and convince yourself that this ring is a field before proceeding.)

Section 7.6: **5(a, b)**

Section 8.1: **8(a)**, **10**

Exercises not from the text: (to be handed in):

Recall that we defined localization in general for any multiplicative system S of a commutative ring R, maybe containing zerodivisors.

1. Let R be a commutative ring and let S be a multiplicative system in R. Let RS^{-1} be the localicalization of R at S. Let I be an ideal of R.

(a). Show that

$$IS^{-1} = \big\{ \frac{r}{s} \mid r \in I, s \in S \big\}$$

is an ideal of RS^{-1}. Show also that $\overline{S} = \{s + I | s \in S\}$ is a multiplicative system in the factor ring R/I. Now show that

$$RS^{-1}/IS^{-1} \cong (R/I)(\overline{S})^{-1}.$$

(b). Let P be a prime ideal and recall that the localization of R at P is $R_P = RS^{-1}$ where $S = \{x \in R | x \notin P\}$. Show that RS^{-1}/PS^{-1} is isomorphic to the field of fractions of the domain R/P.

2. (a). let F be a field. Show that the polynomial ring in infinitely many variables over F, $R = F[x_1, x_2, x_3, \ldots]$, is not a noetherian ring. (Recall that elements of this ring are finite linear combinations with F-coefficients of *monomials* of the form $x_{i_1}^{e_1} x_{i_2}^{e_2} \ldots x_{i_m}^{e_m}$, for any $1 \leq i_1 < i_2 < \cdots < i_m$ and any $e_j \geq 1$.)

(b). Show that if S is a noetherian ring, then so is S/I, for any ideal I. Thus the noetherian property passes to factor rings.

(c). Give an example of a ring S and a subring $R \subseteq S$ such that S is noetherian, but R is not. Thus the noetherian property does not pass to subrings. (Hint: take an example of a non-noetherian ring and embed it in a ring that is noetherian for trivial reasons.)