1. Recall that the center of a group $G$ is $Z = Z(G) = \{a \in G | ab = ba \text{ for all } b \in G\}$, which is always a normal subgroup of $G$.

Prove that if $G/Z$ is a cyclic group, then $G = Z$ and $G$ is abelian.

2. Suppose that $H$ and $K$ are normal subgroups of $G$ such that $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H, k \in K$.

3. Recall that given a subset $X$ of a group $G$, the subgroup of $G$ generated by $X$, written $\langle X \rangle$, is the unique smallest subgroup of $G$ containing $X$ (equivalently, the intersection of all subgroups of $G$ which contain $X$). The group $G$ is finitely generated if $G = \langle X \rangle$ for a finite set $X$.

Prove that $(\mathbb{Q}, +)$ is not finitely generated, where $\mathbb{Q}$ is the rational numbers.

4. Consider the dihedral group $D_{2n}$ of order $2n$ for some $n \geq 3$. We defined this in class as a subgroup of invertible $2 \times 2$ real matrices, but in this problem it is most convenient to work with its presentation as follows: $D_{2n}$ is generated by elements $a, b$ such that $D_{2n} = \{a^i b^j | 0 \leq i \leq n - 1, 0 \leq j \leq 1\}$, where $a^n = 1, b^2 = 1, ba^i = a^{-i}b$ for all $i$. (We will consider presentations more formally later). (Note: a previously posted version of this problem allowed $n \geq 2$, but part (b) below doesn’t work for $n = 2$, so just consider $n \geq 3$.)

(a). Find the center $Z$ of $D_{2n}$ for each $n \geq 3$.
(b). Show that if $n$ is even, then $D_{2n}/Z \cong D_n$.

5. Recall the definition of the direct product $G_1 \times G_2$ of two groups $G_1$ and $G_2$ (Section 1.1 of the text).

Let $H$ and $K$ be normal subgroups of a group $G$ such that $G = HK$. Prove that $G/(H \cap K) \cong (G/H) \times (G/K)$.

6. Let $G$ be a finite group with normal subgroup $N$. Let $H \leq G$ be another subgroup of $G$.

Show that if $|H|$ and $|G : N|$ are relatively prime, then $H \subseteq N$. Conclude that if $|N|$ and $|G : N|$ are relatively prime, then $N$ is the unique subgroup of $G$ of order $|N|$. (Hint: consider $HN$.)
7. Let \( p \) be a prime and let \( G \) be a group of order \( p^a m \), where \( \gcd(m, p) = 1 \). Let \( P \) be a subgroup of \( G \) of order \( p^a \) (later we will call \( P \) a Sylow \( p \)-subgroup). Let \( N \) be any normal subgroup of \( G \), say of order \( |N| = p^b n \) where \( \gcd(n, p) = 1 \). Show that \( |P \cap N| = p^b \) and \( |PN/N| = p^{a-b} \).

8. (a). Suppose that \( G \) is a group. Show that one cannot have \( G = H_1 \cup H_2 \), where each \( H_i \) is a proper subgroup of \( G \) (that is, not equal to all of \( G \)).

(b). Suppose that \( G \) is a finite group and that \( G = H_1 \cup H_2 \cup H_3 \), where each \( H_i \) is a proper subgroup of \( G \). Show that \( |G : H_i| = 2 \) for all \( i \). Also, find an example where this actually happens. (Hint: first show by counting that at least one of the subgroups, say \( H_1 \), has index 2. Then prove that this forces \( H_1H_i = G \) and \( |H_i : H_1 \cap H_i| = 2 \), for \( i = 2, 3 \).)