1. Suppose that $H \leq G$ where $|G| = 12$ and $|H| = 3$. Prove that either $H$ is normal in $G$ or else $G \cong A_4$. (Hint: use a left coset action.)

2. Prove that if $H$ is a subgroup of $G$ such that $|G : H| = n < \infty$, then there is a normal subgroup $K \subseteq G$ with $K \leq H$ such that $|G : K| \leq n!$.

3. Assume that the finite group $G$ acts transitively on the finite set $X$ and let $H$ be a normal subgroup of $G$. Then $H$ also acts on $X$ (by restricting the action to $H$), but this action might no longer be transitive. Let $O_1, \ldots, O_r$ be the orbits of the action of $H$ on $X$.

(a). For each $g \in G$ and $1 \leq i \leq r$ define $gO_i = \{gx | x \in O_i\}$. Show that $gO_i = O_j$ for some $j$. Prove that the rule $gO_i = O_j$ defines an action of the group $G$ on the set $Y = \{O_1, \ldots, O_r\}$. Prove that this is a transitive action on $Y$, and conclude that $|O_i| = |O_j|$ for all $i, j$.

(b). Prove that if $x \in O_1$ then $|O_1| = |H : H \cap G_x|$ and that $r = |G : HG_x|$. (Recall that $G_x$ means the stabilizer of $x$ under the action of $G$ on $X$, namely $G_x = \{g \in G | gx = x\}$).

4. Let $\mathcal{K}$ be a conjugacy class in $S_n$ for some $n \geq 3$, that is, an orbit of the action of $S_n$ on itself by conjugation. Restricting this action we obtain an action of $A_n$ on $S_n$ by conjugation.

(a). Show that either (i) $\mathcal{K}$ is also an orbit of the $A_n$-action on $S_n$, or else (ii) $\mathcal{K}$ is the union of two orbits of equal size under the $A_n$-action. (Hint: problem 3).

(b). Recall that the conjugacy classes of $S_n$ are determined by the possible cycle shapes, so that there is some list of positive integers $k_1, k_2, \ldots k_d$ (possibly with repeats) such that $\mathcal{K}$ consists of all permutations whose disjoint cycle representation has cycles of lengths $k_1, k_2, \ldots, k_d$ (including 1-cycles). Show that case (ii) occurs in part (a) if and only if the list $\{k_i\}$ consists of odd integers without repeats.

5. Let $G$ be a finite group and let $H$ be a proper subgroup of $G$ (that is, $H \neq G$). Recall that a subgroup of the form $gHg^{-1}$ is called a conjugate of $H$.

(a). Show that the number of distinct conjugates of $H$ is equal to $|G : N_G(H)|$, where $N_G(H)$ is the normalizer of $H$ in $G$. (Hint: define a conjugation action of $G$ on the set of all subgroups of $G$).
(b). Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$. Thus a finite group cannot be equal to the union of the conjugates of a proper subgroup. (Hint: show that the union on the right hand side cannot have enough elements).

6. Consider the symmetric group $G = S_p$ for some prime number $p$.
   (a). Calculate the number of distinct subgroups $H$ of $G$ such that $|H| = p$. Show that any two of these subgroups are conjugate to each other.
   (b). Let $H$ be a subgroup of $S_p$ such that $|H| = p$. Show that $|N_G(H)| = p(p - 1)$.

7. We say that a subgroup $H \leq G$ is characteristic in $G$, and write $H \text{ char } G$, if for all automorphisms $\phi$ of $G$, $\phi(H) = H$.
   (a). Show that if $H \text{ char } G$, then $H \trianglelefteq G$.
   (b). Let $H < K < G$, where $H \text{ char } K$ and $K \trianglelefteq G$. Show $H \trianglelefteq G$.
   (c). Show that if $K$ is a cyclic subgroup of $G$ and $K \trianglelefteq G$, then every subgroup $H$ of $K$ satisfies $H \trianglelefteq G$.

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Additional exercise (Do not write up and hand in):
I include this exercise because it gives an interesting and elegant proof of Cauchy’s theorem which you may enjoy working through. I think it is a bit tedious to carefully write up all of the parts and so it is not part of the assigned exercises which you should write up and hand in. We will also see a different proof of Cauchy’s theorem in class.

A. In this exercise you will prove the following theorem often called Cauchy’s theorem:

**Theorem 0.1** If $G$ is a finite group and $p$ is a prime dividing the order of $G$, then $G$ has an element $x$ of order $p$.

Fix a finite group $G$ and a prime $p$ dividing $|G|$. Let

$$X = \{(a_1, a_2, \ldots, a_p) \in G^p \mid a_1a_2\ldots a_p = e\}.$$  

In words, $X$ is the set of (ordered) $p$-tuples of elements of $G$ whose product in the given order is the identity element.

(a). Show that if $a_1a_2\ldots a_p = e$ in a group $G$, then $a_2a_3\ldots, a_pa_1 = e$ as well.
(b). Using (a), show that there is an action of the cyclic group $K = (\mathbb{Z}_p, +)$ on $X$ such that the generator $1$ of $K$ acts on $(a_1, a_2, \ldots, a_p)$ and sends it to $(a_2, a_3, \ldots, a_p, a_1)$.
(c). Show that the orbits of size 1 under the action in (b) are precisely the $p$-tuples $(a, a, \ldots, a)$ where $a^p = e$.
(d). Show that $|X| = |G|^{p-1}$.
(e). Show that there must be more than one orbit of size 1 under the action in (b), and thus there is a nonidentity element $a$ such that $a^p = e$. Conclude that $a$ is an element of order $p$ in $G$, as desired.