Math 200a Fall 2014 Homework 3

Due Friday 10/31/2014 by 3pm in homework box in basement

1. Let \( n \geq 5 \).
   (a). Prove that the only normal subgroups of \( S_n \) are \( \{e\}, A_n \), and \( S_n \).
   (b). Prove that if \( H \leq S_n \) with \( |S_n : H| = d \), where \( 1 < d < n \), then \( H = A_n \).

2. Let \( |G| = pqr \) for some distinct primes \( p, q, r \) with \( p < q < r \).
   Prove that \( G \) has a normal Sylow subgroup of order \( p \), \( q \), or \( r \).

3. Let \( |G| = 595 = (5)(7)(17) \). Show that all Sylow subgroups of \( G \) are normal.

4. Suppose that \( |G| = 231 = (3)(7)(11) \). Show that \( G \) has a normal Sylow 11-subgroup \( P \) and prove that \( P \subseteq Z(G) \). (Hint: If \( Q \) is a Sylow 7-subgroup and \( R \) is a Sylow 3-subgroup, show that \( PQ \) and \( PR \) are Abelian).

5. Let \( G \) be a finite group with subgroups \( P \leq H \leq K \leq G \), where \( P \) is a Sylow \( p \)-subgroup of \( G \).
   (a). Prove that if \( P \leq H \) and \( H \leq K \), then \( P \leq K \).
   (b). Prove that \( N_G(N_G(P)) = N_G(P) \).

6. Let \( P \) be a Sylow \( p \)-subgroup of the finite group \( G \). Let \( H \leq G \) be a subgroup of \( G \).
   (a). Show that there exists \( g \in G \) such that \( gPg^{-1} \cap H \) is a Sylow \( p \)-subgroup of \( H \).
   (b). Suppose that \( H \leq G \). Prove that \( P \cap H \) is a Sylow \( p \)-subgroup of \( H \).
   (c). Suppose that \( P \leq G \). Prove that \( P \cap H \) is a Sylow \( p \)-subgroup of \( H \), and is the unique Sylow \( p \)-subgroup of \( H \).
   (d). If neither \( P \) nor \( H \) is normal in \( G \), show that \( P \cap H \) need not be a Sylow \( p \)-subgroup of \( H \) in general.

7. Let \( |G| = p(p + 1) \) where \( p \) is prime. Show that \( G \) has either a normal subgroup of order \( p \) or a normal subgroup of order \( p + 1 \). (Hint: If \( n_p > 1 \), choose \( x \in G \) of order not equal to 1 or \( p \). Study the conjugacy class of \( x \) and \( |C_G(x)| \)).