Math 200a Fall 2014 Homework 4

Due *Monday* 11/10/2014 by 3pm in homework box in basement.

Note: I have given a little extra time for this homework since these problems are lengthy and the material on presentations will not be covered in class until Monday November 3. Recall also there is no homework due on Friday 11/15 as that is the day of the midterm. Problems about solvable and nilpotent groups will appear on the next homework due after the exam. I will not ask questions about solvable and nilpotent groups on the midterm.

1. While a semidirect product $G = H \rtimes_\psi K$ depends in general on the choice of homomorphism $\psi : K \to \mathrm{Aut}(H)$, sometimes different choices of ψ lead to isomorphic semidirect products. This problem explores some cases where this happens.

(a). Suppose that $\theta \in \mathrm{Aut}(H)$ and let $\phi_\theta : \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ be the inner automorphism of $\mathrm{Aut}(H)$ given by $\rho \mapsto \theta\rho\theta^{-1}$. Let $\psi_2 = \phi_\theta \circ \psi : K \to \mathrm{Aut}(H)$. Prove that $H \rtimes_\psi K$ and $H \rtimes_{\psi_2} K$ are isomorphic groups. (Hint: Try the map $H \rtimes_\psi K \to H \rtimes_{\psi_2} K$ given by $(h, k) \mapsto (\theta(h), k)$.)

(b) Suppose that $\rho : K \to K$ is an automorphism of K and define $\psi_2 = \psi \circ \rho : K \to \mathrm{Aut}(H)$. Prove that $H \rtimes_\psi K$ and $H \rtimes_{\psi_2} K$ are isomorphic groups.

2. Suppose that p and q are primes with $p < q$ where p divides $q - 1$. Show that there are precisely two groups of order pq up to isomorphism.

3. Classify groups G of order 20 up to isomorphism (there are 5 such groups). Be sure to prove the 5 examples you claim are different really are pairwise non-isomorphic.

4. Find a presentation for each of the three non-Abelian groups you found in problem 3 (one for each isomorphism type). Carefully justify that your presentations are correct.

5. Classify groups of order 75 up to isomorphism. (Hint: Find the order of the group $\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5)$ and show that all subgroups of order 3 in this group are conjugate. You don't need to find any of the elements of order 3 explicitly.)

6. We will show in class (Monday 11/3) that the dihedral group D_{2n} has the presentation

$$D_{2n} \cong \langle a, b | a^n = e, b^2 = e, ba = a^{-1}b \rangle.$$

Using this presentation, show that the automorphism group $\mathrm{Aut}(D_{2n})$ is isomorphic to a semidirect product $\mathbb{Z}_n \rtimes_\psi (\mathbb{Z}_n)^*$ for some map $\psi : (\mathbb{Z}_n)^* \to \mathrm{Aut}(\mathbb{Z}_n)$. Also, find ψ explicitly.

(Hint: an automorphism of D_{2n} is determined by where a and b go. Show that the automorphisms are given by mapping a to any other generator of the rotation subgroup $\langle a \rangle$ and mapping b to any other reflection.)

7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the formulas $f(x) = -x$ and $g(x) = x+1$. Let $G = \langle f, g \rangle$ be the subgroup of the group $S_{\mathbb{R}}$ of all permutations of the set $\mathbb{R}$ which is generated by f and g. Prove carefully that

$$G \cong \langle a, b \,|\, b^2 = e, ba = a^{-1}b \rangle.$$

(This group is called the *infinite dihedral group*, D_∞.)