Math 200a Fall 2014 Homework 5

Due Friday 11/21/2014 by 3pm in homework box in basement.

1. Let G be a group (do not assume that G is finite).
(a). Show that if G is nilpotent, then all subgroups and quotient groups of G are also nilpotent.
(b). Show that if $G/Z(G)$ is nilpotent, then G is nilpotent.

2. (a). Prove that a finite group G is nilpotent if and only if whenever $a, b \in G$ are elements with relatively prime orders, then a and b commute.
(b). Prove that the dihedral group D_{2n} is nilpotent if and only if n is a power of 2. (one way is to use part (a)).

3. Suppose that G is finite and has the property that every maximal subgroup of G has prime index. Prove that G is solvable, in the following steps.
(a). Prove that if P is a Sylow p-subgroup of G and $N_G(P) \leq H \leq G$ for some subgroup H, then $|G : H| \equiv 1 (\operatorname{mod} p)$. (This part is true in any finite group. Hint: P is a Sylow p-subgroup of H and $N_G(P) = N_H(P)$.)
(b). Taking p to be the largest prime dividing the order of the group G, show that G has a normal Sylow p-subgroup.
(c). Conclude the proof by induction on the order.

4. Let G be a finite group. The *Frattini subgroup* of a group G, denoted $\Phi(G)$, is the intersection of all of the maximal subgroups of G.
(a). Prove that $\Phi(G)$ is a characteristic subgroup of G.
(b). Prove that $\Phi(G)$ is a nilpotent group. (Hint: use Frattini's argument, p. 193 in the text).
(c). Now let $G = P$ be a p-group for some prime p. Recall that an *elementary abelian p-group* is a finite direct product of copies of $\mathbb{Z}_p$. Show that $P/\Phi(P)$ is an elementary abelian p-group, and that $\Phi(P)$ is the unique smallest normal subgroup with this property, i.e. if N is any normal subgroup of P such that P/N is elementary abelian, then $\Phi(P) \subseteq N$.

5. An element x of a ring R is called *nilpotent* if $x^n = 0$ for some $n \geq 1$. Let R be a commutative ring.

(a). Prove that if $x \in R$ is nilpotent and $r \in R$, then rx is nilpotent.

(b). Prove that if x and y are both nilpotent, then $x+y$ is nilpotent. (Hint: the binomial formula is valid in any commutative ring).

(c). Prove that the set of nilpotent elements in the ring R forms an ideal I. (I is called the *nilradical* of R).

(d). Prove that if u is any unit in R, and x is nilpotent, then $u+x$ is again a unit in R.

6. Let R be a commutative ring. The ring $R[[x]]$ of *formal power series* in one variable is the ring whose elements are formal sums $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ for some $a_n \in R$. (Note that any choice of coefficients $a_n \in R$ defines a power series, and two power series are equal by definition if and only if they have the same coefficients. "Convergence" of the power series is meaningless in this general context.) Addition and multiplication of power series are defined analogously as for polynomials. (See Section 7.2 exercise 3). You should convince yourself that $R[[x]]$ satisfies the axioms of a ring.

(a). Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in the ring $R[[x]]$ if and only if a_0 is a unit in R.

(b). Prove that if R is a domain, then $R[[x]]$ is a domain.

(c). Suppose that R is a field. Show that the set of power series in $R[[x]]$ whose constant term is 0 is a maximal ideal I of $R[[x]]$. Prove that I is the unique maximal ideal of R. (remark: a ring with a unique maximal ideal is called *local*.)