Math 200a Fall 2011 Final Exam

December 7, 2011

The exam is designed to be too long for most students to finish. Concentrate on doing several problems very well rather than giving half-solutions to all problems. You may use major theorems proved in class or the textbook, unless the point of the problem is to repeat the proof of such a result. Avoid quoting results proved only in homework exercises.

NAME:

Problem 1/ 10	
Problem 2 /20	
Problem 3 /10	
Problem 4 /10	
Problem 5 /10	
Problem 6 /10	
Problem 7 /10	
Total /80	

1. (10 pts) State the class equation for a finite group. Then use it to show that every finite p-group, where p is a prime, has a nontrivial center. (This is a theorem from class and the text, so you are expected to repeat the proof of it here.)

2. (a). (7 pts) Show that every group G of order 70 is isomorphic to a semidirect product $\mathbb{Z}_{35} \rtimes_\phi \mathbb{Z}_2$.

(b). (8 pts) Find all of the possible homomorphisms $\phi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_{35})$ occurring in part (a), and show that there are exactly 4 distinct groups of order 70 up to isomorphism.

(c). (5 pts) Is every group of order 70 solvable? Is every group of order 70 nilpotent? Justify your answers. (This part may be doable even if you did not finish parts (a) and (b)).

3. (a). (3 pts) Let $p \geq 3$ be a prime. Find an explicit Sylow-p-subgroup P of the symmetric group $G = S_{2p}$.

(b). (7 pts) Let $N_G(P)$ be the normalizer of P in G. Give a formula relating the number $|N_G(P)|$ and the number of distinct Sylow p-subgroups of $G = S_{2p}$. Then calculate one of these numbers directly, and hence find both numbers.

4. (10 pts) State Zorn's lemma. Then use it to show that if R is a ring with identity, then any proper ideal I of R is contained in a maximal ideal. (This is a theorem from class and the text, so you are expected to repeat the proof.)

5. An integral domain R is *Bezout* if given any elements $a, b \in R$, one has $(a, b) = (d)$ for some $d \in R$. An integral domain R is called 2-*stage Euclidean* if there is a function $N : R \to \{0, 1, 2, 3 \dots\}$ with the property that given any $a, b \in R$ with $b \neq 0$, there are elements $q, r, q', r' \in R$ with $a = qb + r$ and $b = q'r + r'$, and where either $r' = 0$ or $N(r') < N(b)$.

(a). (7 pts) Prove that a 2-stage Euclidean domain is Bezout.

(b). (3 pts) Prove that a noetherian 2-stage Euclidean domain is a PID.

6. (10 pts) Let R be a noetherian ring, and let S be any multiplicative system in S. Prove that the localization RS^{-1} is also a noetherian ring.

7. (10 pts) Consider the ring $\mathbb{Q}[x, y]/(y^2+x+1)$. Is it an integral domain? Is it a field? Justify your answers.