# MATH 200A FALL 2016 FINAL EXAM

**Instructions:** Do as many of the problems as well as you can; the exam may be too long for you to finish. You may use major theorems proved in class or the textbook, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.

**1 (10 pts).**

(a). Let $H$ be a subgroup of $G$. Show that the number of distinct conjugates of $H$ in $G$ is $|G : N_G(H)|$.

(b). Show that if $G$ is a nontrivial finite group and $H$ is a proper subgroup of $G$, then $G \neq \bigcup_{g \in G} gHg^{-1}$.

**2 (15 pts).**

Let $|G| = 105 = (3)(5)(7)$.

(a). Show that either $G$ has a normal Sylow 5 subgroup or a normal Sylow 7 subgroup. Conclude that $G$ has a subgroup $H$ with $|H| = 35$.

(b). Show that $G$ is isomorphic to a semidirect product $\mathbb{Z}_{35} \rtimes_\psi \mathbb{Z}_3$. Show that there are only two such groups $G$ up to isomorphism.

(c). Find a presentation of the non-Abelian group occurring in part (b). Justify that your presentation is correct.

**3 (10 pts).**

(a). Let $H \subseteq K \subseteq G$ where $H$ and $K$ are subgroups of the group $G$. Suppose that $H \operatorname{char} K$ and $K \trianglelefteq G$. Prove that $H \trianglelefteq G$.

(b). Let $G$ be a finite solvable group. Let $K$ be a minimal normal subgroup of $G$ (so there are no nontrivial normal subgroups of $G$ strictly contained in $K$). Prove that $K$ is elementary Abelian, that is, $K \cong (\mathbb{Z}_p)^{\times n}$ is a direct product of finitely many copies of $\mathbb{Z}_p$ for some prime $p$ and $n \geq 1$.

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**4 (10 pts).**

(a). State Zorn's Lemma.

(b). Let $R$ be a commutative ring. Show that $R$ has a prime ideal $P$ which is minimal under inclusion in the set of prime ideals.

**5 (15 pts).**

(a). Prove that any Euclidean domain is a PID.

(b). Show that $R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} | a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the norm function $N(a + b\sqrt{-2}) = a^2 + 2b^2$, and conclude that $R$ is PID.

(c). Find a factorization of 15 into irreducibles in $R$. Justify your answer.

**6 (10 pts).** Let $R$ be a commutative ring, and let $X$ be a multiplicative system in $R$. Let $S = RX^{-1}$ be the localization of $R$ along $X$.

(a). Let $J$ be an ideal of $S$. Show that $I = \left\{ a \in R \middle| \frac{a}{1} \in J \right\}$ is an ideal of $R$, and that $J = \left\{ \frac{a}{x} \middle| a \in I, x \in X \right\}$.

(b). Prove that if $R$ is a Noetherian ring, then $RX^{-1}$ is Noetherian also.