MATH 200A FALL 2016 MIDTERM

Instructions: Do as many of the problems as well as you can; the exam may be too long for you to finish. You may use major theorems proved in class or the textbook, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem. (For example, you could quote the result that if H char K and $K \trianglelefteq G$, then $H \trianglelefteq G$, which is a fundamental result you proved in an exercise). You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.

1 (10 pts).
(a) Prove that if G is a group such that $G/Z(G)$ is cyclic, then $Z(G) = G$. (This was a homework exercise; I want you to reprove it).
(b) Show that if G is a group of order p^3 for some prime p, then either G is Abelian or else $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

2 (15 pts).
Let G be a group of order 30.
(a) Prove that G has either a normal Sylow 5-subgroup or a normal Sylow 3-subgroup.
(b) Improve part (a) by showing that G has both a normal Sylow 5-subgroup and a normal Sylow 3-subgroup.
(c) Show that there are at least two different (i.e. non-isomorphic) non-Abelian groups of order 30.

3 (10 pts).
Let G act on a set X. For any subset $Y \subseteq X$ and $g \in G$, let $gY = \{g \cdot y \mid y \in Y\}$. A *block* is a subset Y of X with the property that for all $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$.

Suppose that G acts transitively on X. Pick $x \in X$ and let G_x be the stabilizer of x. Suppose that there exists a block Y with $|Y| > 1$, $Y \subsetneq X$, and $x \in Y$. Show that there exists a subgroup H of G with $G_x \subsetneq H \subsetneq G$.

Date: October 28, 2016.