Math 200a Fall 2021 Homework 1

Due Friday 10/1/2021 in class

1. Let G be a set with a binary operation $*$ such that

(i) $*$ is associative.

(ii) There exists $e \in G$ such that $e * a = a$ for all $a \in G$.

(iii) For all $a \in G$, there exists $b \in G$ such that $b * a = e$.

In other words, one only assumes that e is a left identity and that left inverses exist. Show that G is a group, i.e. these seemingly weaker axioms are actually equivalent to the standard definition of a group.

2. Let G be a group. Suppose that given any triple of elements of G, some pair of the elements commute with each other. Show that G is abelian.

3. Suppose that G is a group for which $a^2 = 1$ for all $a \in G$. Show that G is abelian.

4. Let M be a monoid with identity element 1. We showed in class that

$$G(M) = \{a \in M | \text{there exists } b \in M \text{ such that } ba = 1 = ab\}$$

is a group.

(a) Exercise 1 suggests that perhaps

$$G'(M) = \{a \in M | \text{there exists } b \in M \text{ such that } ba = 1\}$$

is a group for any monoid M. Is this true?

(b) Find an example of an infinite monoid M such that the associated group $G(M) = \{1\}$ is trivial.

5. Let G be a group. Given $a \in G$, define $\ell_a : G \to G$ by $\ell_a(b) = ab$ for all $b \in G$. In other words, ℓ_a is the "left multiplication by a function". Similarly, define $r_a : G \to G$ by $r_a(b) = ba$ for all $b \in G$.

(a). Show that ℓ_a and r_a are in the group $\mathrm{Sym}(G)$ for all $a \in G$.

(b). Let $L = \{\ell_a | a \in G\}$ and $R = \{r_a | a \in G\}$. Show that L and R are both subgroups of $\mathrm{Sym}(G)$ and that $L \cong G \cong R$.

(c). Show that $C_{\mathrm{Sym}(G)}(L) = R$ and $C_{\mathrm{Sym}(G)}(R) = L$. (Recall that $C_H(X)$ means the centralizer of the subset X in a group H, namely $\{a \in H | ax = xa \text{ for all } x \in X\}$).

6. Let G be a group. Let H, K, and L be subgroups of G such that $H \subseteq K$. Prove that $K \cap HL = H(K \cap L)$.