Math 200a Fall 2021 Homework 2

Due Friday 10/8/2021 in class

1. Let $Z = Z(G)$ be the center of a group G.

(a) Suppose that G/Z is cyclic. Prove that G is abelian and hence G/Z is actually trivial.

(b) Let G be a group. Show that $\mathrm{Inn}(G)$ cannot be a nontrivial cyclic group.

2. Let H_1, H_2, H_3 be subgroups of a group G.

(a) If $G = H_1 \cup H_2$, then $G = H_1$ or $G = H_2$.

(b) If G is finite and $G = H_1 \cup H_2 \cup H_3$, then either $G = H_i$ for some i, or $|G : H_i| = 2$ for all i. (Hint: one of the H_i has more than $1/3$ of the elements of G.)

(c) Give an example showing that the second case in (b) can occur, that is where G is the union of three proper subgroups all of index 2.

3. Let G be a finite cyclic group of order n, say with $G = \langle a \rangle$.

(a) Show that for any $0 \leq i < n$, the function $\phi_i : G \to G$ defined by $\phi_i(a^j) = a^{ij}$ is a homomorphism.

(b) Show that every homomorphism $f : G \to G$ is equal to ϕ_i for some i.

(c) Show that ϕ_i is an automorphism of G if and only if $\gcd(i, n) = 1$.

(d) Prove that $\mathrm{Aut}(G)$ is isomorphic as a group to U_n, the group of units of intgers mod n under multiplication.

4. Let G be a group. A subgroup M of G is *maximal* if $M \neq G$ and whenever H is a subgroup of G with $M \leq H \leq G$, then either $H = M$ or $H = G$.

(a) If G is finite, show that every proper subgroup of G is contained in a maximal subgroup.

(b) Suppose that G is finite and contains precisely one maximal subgroup. Show that G is cyclic of order p^n for some prime p and $n \geq 1$.

(c) Suppose that the trivial subgroup $\{1\}$ is a maximal subgroup of G. Prove that G is cyclic of prime order.

5. Let $G = \{f : \mathbb{R} \to \mathbb{R} | f = ax + b \text{ for some } a, b \in \mathbb{R}, a \neq 0\}$.

(a) Show that G is a group under composition. (The elements of G are called *affine transformations* of the real line).

(b) Show that $H = \{f : \mathbb{R} \to \mathbb{R} | f = x + b \text{ for some } b \in \mathbb{Z}\}$ is a subgroup of G.

(c) Show that there is $g \in G$ such that $gHg^{-1} \subsetneq H$.

(d) Recall that the normalizer of H is defined to be $N_G(H) = \{g \in G | gHg^{-1} = H\}$. Observe on the other hand that $\{g \in G | gHg^{-1} \subseteq H\}$ is not even a subgroup of G. This example shows why we want to define the normalizer using $=$ and not $\subseteq$ in general.

6. Consider the group $G = (\mathbb{Q}/\mathbb{Z}, +)$. Fix a prime p.

(a) Let $H = \{a \in G | p^n a = 0 \text{ for some } n \geq 0\}$. Show that H is a subgroup of G.

(b) Show that H has a subgroup H_i of order p^i for each $i \geq 0$, where $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$, and that the H_i are the only proper subgroups of H.

(c) Prove that H is a group with no maximal subgroups.

7. Let G be a finite group. A subgroup K of G satisfying $\gcd(|K|, |G : K|) = 1$ is called a *Hall subgroup* of G.

(a) Let $N \trianglelefteq G$ and $H \leq G$. Show that if $\gcd(|H|, |G : N|) = 1$ then $H \subseteq N$. (Hint: Consider HN.)

(b) Suppose that $N \trianglelefteq G$ and N is a Hall subgroup of G. Prove that N is the unique subgroup of G with order $|N|$.

(c) Let H be a Hall subgroup of G and let $N \trianglelefteq G$. Show that $H \cap N$ is a Hall subgroup of N, and that HN/N is a Hall subgroup of G/N.