Math 200a Fall 2021 Homework 3

Due Friday 10/15/2021 in class

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the formulas $f(x) = -x$ and $g(x) = x+1$. Let $G = \langle f, g \rangle$ be the subgroup of $\mathrm{Sym}(\mathbb{R})$ (the group of all permutations of the set $\mathbb{R}$) generated by f and g. Prove carefully that $G \cong \langle a, b \,|\, b^2 = 1, ba = a^{-1}b \rangle$.

2. Let $G = F(a, b)$ be a free group in two variables. Let H be the subgroup of G given by $H = \langle b, a^{-1}ba, a^{-2}ba^2, \dots \rangle$. Show that H is free on the subset $\{b, a^{-1}ba, a^{-2}ba^2, \dots\}$.

(Hint: Let $G' = F(w_0, w_1, w_2 \dots)$ be a free group on countably many variables and let $\phi : G' \to G$ be the homomorphism with $w_i \mapsto a^{-i}ba^i$. Show that ϕ is injective by proving that no nontrivial reduced word in the w_i can map to 1 under ϕ.

3. Presentations are useful for finding automorphisms of groups. Let $n \geq 3$ be fixed. As shown in class, the dihedral group D_{2n} has the presentation $\langle a, b | a^n = 1, b^2 = 1, ba = a^{-1}b \rangle$. Think of D_{2n} as this presented group.

(a) Show that any automorphism $\sigma : D_{2n} \to D_{2n}$ must satisfy $\sigma(a) = a^i$, where $0 \leq i \leq n-1$ with $\gcd(i, n) = 1$ and $\sigma(b) = a^j b$ for some $0 \leq j \leq n-1$.

(b) Show that given any i, j satisfying $0 \leq i, j \leq n-1$ and $\gcd(i, n) = 1$, there is a unique automorphism σ of D_{2n} with $\sigma(a) = a^i$ and $\sigma(b) = a^j b$. Conclude that $|\operatorname{Aut}(D_{2n})| = n\varphi(n)$, where φ is the Euler φ-function.

4. A *generalized quaternion group* of order $4n$ is defined to be the group given by the presentation

$$Q_{4n} = \langle x, y | x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle.$$

(a) Let $\zeta \in \mathbb{C}$ be a primitive $2n$th root of unity. Prove that Q_{4n} is isomorphic to the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by the matrices

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(b) Prove that when $n = 2$, Q_8 as defined above is a presentation of the usual quaternion group of order 8.

5. Assume that the group G acts transitively on a set X and let H be a normal subgroup of G. Then H also acts on X (by restricting the action to H), but this action might no longer be transitive. Let $Y = \{\mathcal{O}_\alpha\}_{\alpha \in I}$ be the set of orbits of the action of H on X, where α ranges over some index set I.

(a) For each $g \in G$ and orbit $\mathcal{O}_\alpha$ define $g \cdot \mathcal{O}_\alpha = \{g \cdot x | x \in \mathcal{O}_\alpha\}$. Show that $g \cdot \mathcal{O}_\alpha = \mathcal{O}_{\alpha'}$ for some α'. Prove that the rule $g \cdot \mathcal{O}_\alpha = \mathcal{O}_{\alpha'}$ defines an action of the group G on the set Y. Prove that this is a transitive action on Y, and conclude that $|\mathcal{O}_\alpha| = |\mathcal{O}_{\alpha'}|$ for any $\alpha, \alpha' \in I$.

(b) Suppose that X is finite. Prove that if $x \in \mathcal{O}_\alpha$ then $|\mathcal{O}_\alpha| = |H : H \cap G_x|$ and that $|Y| = |G : G_x H|$. Do not assume that G is finite.

6. Let G be a finite group and let H be a proper subgroup of G (that is, $H \neq G$). Recall that a subgroup of the form gHg^{-1} is called a *conjugate* of H.

(a). Show that the number of distinct conjugates of H is equal to $|G : N_G(H)|$, where $N_G(H)$ is the normalizer of H in G. (Hint: apply orbit/stabilizer to an appropriate action).

(b). Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$. Thus a finite group cannot be equal to the union of the conjugates of a proper subgroup. (Hint: show that the union on the right hand side cannot have enough elements).