Math 200a Fall 2021 Homework 4

Due Friday 10/22/2021 in class

1. let H and K be subgroups of a group G. Given $g \in G$, $HgK = \{hgk | h \in H, k \in K\}$ is called an (H,K)-*double coset of* G.

(a) Prove that for $g_1, g_2 \in G$, either $Hg_1K = Hg_2K$ or else $Hg_1K \cap Hg_2K = \emptyset$. In other words, any two distinct double cosets are disjoint.

(b) Assume that $|H| < \infty$ and $|K| < \infty$. Show that $|HgK| = |H||K|/|H \cap gKg^{-1}|$ for all $g \in G$.
(Hint: for both parts, study the orbits of a suitable action).

2. In class we proved Cauchy's theorem that if p is a prime dividing $|G|$, then G has an element of order p. Our proof relied on induction on the order of group and the class equation. The following exercise gives an interesting combinatorial proof that uses only the basic properties of actions. Let

$$X = \{(a_1, a_2, \dots, a_p) \in G^p \mid a_1 a_2 \dots a_p = 1\}.$$

In words, X is the set of (ordered) p-tuples of elements of G whose product in the given order is the identity.

(a) Show that if $a_1 a_2 \dots a_p = 1$, then $a_2 a_3 \dots a_p a_1 = 1$ as well.

(b) Let $K = \langle k \rangle$ be a cyclic group of order p, generated by some k. Show that there is an action of K on X such that $k \cdot (a_1, a_2, \dots, a_p) = (a_2, a_3, \dots, a_p, a_1)$.

(c) Show that the orbits of size 1 in X under the action in (b) are precisely the p-tuples $(a, a, \dots, a)$ where $a^p = 1$.

(d) Show that $|X| = |G|^{p-1}$.

(e) Show that there must be more than one orbit of size 1 under the action in (b), and thus there is a nonidentity element g such that $g^p = 1$. Thus $|g| = p$ as desired.

3. Let $|G| = p(p+1)$ where p is prime. Show that G has either a normal subgroup of order p or a normal subgroup of order $p+1$. (Hint: If $n_p > 1$, choose $x \in G$ of order not equal to 1 or p. Study the conjugacy class of x and $|C_G(x)|$.)

4. Let $|G| = pqr$ for some distinct primes p, q, r with $p < q < r$.
Prove that G has a normal Sylow subgroup of order p, q, or r.

5. Let $|G| = p^2q^2$ for primes $p < q$. Show that G is not a simple group. In fact, if $|G| \neq 36$ then show that G has a normal p-Sylow subgroup or a normal Sylow q-subgroup.

6. Let G be a finite group with subgroups $P \leq H \leq K \leq G$, where P is a Sylow p-subgroup of G.

(a) Prove that if $P \trianglelefteq H$ and $H \trianglelefteq K$, then $P \trianglelefteq K$.

(b) Prove that $N_G(N_G(P)) = N_G(P)$.

7. Let G be a finite group and fix a prime p. Write $n_p(G)$ for the number of Sylow p-subgroups of the group G.

(a) Let $N \trianglelefteq G$. Let Q be a Sylow p-subgroups of G/N. Show that there is a Sylow p-subgroup P of G such that $Q = PN/N$. Conclude that $n_p(G/N) \leq n_p(G)$.

(b) Let $H \leq G$. Let Q be a Sylow p-subgroup of H. Show that there is a Sylow p-subgroup P of G such that $Q = P \cap H$. Conclude that $n_p(H) \leq n_p(G)$.

(c) Let $H \trianglelefteq G$. Show that if P is a Sylow p-subgroup of G then $P \cap H$ is a Sylow p-subgroup of H.