Math 200a Fall 2021 Homework 5

Due Friday 10/29/2021 in class

1. Let p be a prime. Let $G = S_p$. Let P be a Sylow p-subgroup of G. Show that $|N_G(P)| = p(p-1)$.

2. Consider S_n for $n \geq 5$.

(a) Show that the only normal subgroups of S_n are $\{1\}$, A_n, and S_n.

(b) Let H be a subgroup of S_n with $1 < d = |S_n : H| < n$. Then $d = 2$ and $H = A_n$.

3. Suppose that G is a finite group with $|G| = (2^k)(m)$ where m is odd. Suppose that G has a cyclic Sylow 2-subgroup. Show that G has a unique subgroup H with $|H| = m$.

4. Suppose that G is a simple group with $|G| = 60$.

(a) Show that $n_3 = 10$ and $n_5 = 6$.

(b) Suppose that $n_2 = 15$. Show that there are Sylow 2-subgroups P and Q with $|P \cap Q| = 2$. Show that $|N_G(P \cap Q)| = 12$.

(c) If $n_2 = 5$ then G has a subgroup H with $|H| = 12$ in this case as well.

(d) There is an injective homomorphism of groups $\phi : G \to S_5$.

(e) Conclude that $G \cong A_5$.

(f) In retrospect, is $n_2 = 5$ or is $n_2 = 15$?

5. Let σ be an n-cycle in S_n. Show that the conjugacy class of σ has $(n-1)!$ elements and that $C_{S_n}(\sigma) = \langle \sigma \rangle$.

6. Let $n \geq 3$. In class and the course notes, we showed that if $\sigma \in A_n$, then the conjugacy class $\mathrm{Cl}_{A_n}(\sigma) = \mathrm{Cl}_{S_n}(\sigma)$ is equal to the full S_n-conjugacy class, or else $\mathrm{Cl}_{A_n}(\sigma)$ has precisely half of the elements in $\mathrm{Cl}_{S_n}(\sigma)$. Moreover, we saw that the first case happens when the centralizers satisfy $|C_{A_n}(\sigma)| = |C_{S_n}(\sigma)|/2$, while the second case happens when $C_{A_n}(\sigma) = C_{S_n}(\sigma)$.

Suppose that the disjoint cycle form of σ (with 1-cycles included) consists of cycles of lengths $k_1, k_2, \ldots, k_d$. Show that the second case above happens precisely when the k_i are distinct odd integers.

7. Let H and K be groups and let $G = H \times K$. Identify H and K with subgroups of G as usual.

(a) Suppose that D is a subgroup of G such that $D \cap H = D \cap K = \{1\}$. Prove that there are subgroups $H' \leq H$ and $K' \leq K$ and an isomorphism of groups $\phi : H' \to K'$ such that $D = \{(h, \phi(h)) | h \in H'\}$. In other words, D is the graph of a partial isomorphism from H to K.

(b) If D is as in part (a), show that $D \trianglelefteq G$ if and only if $H' \leq Z(H)$ and $K' \leq Z(K)$.

(c) Suppose that H and K are nonabelian simple groups. Show that the only normal subgroups of G are $\{1\}, H, K$, and G.