Math 200a Fall 2021 Homework 6

Due Friday 11/12/2021 in class

1. While a semidirect product $G = H \rtimes_{\psi} K$ depends in general on the choice of homomorphism $\psi : K \to \mathrm{Aut}(H)$, sometimes different choices of ψ lead to isomorphic semidirect products. This problem explores some cases where this happens.

(a) Suppose that $\theta \in \mathrm{Aut}(H)$ and let $\phi_{\theta} : \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ be the inner automorphism of $\mathrm{Aut}(H)$ given by $\rho \mapsto \theta\rho\theta^{-1}$. Let $\psi_2 = \phi_{\theta} \circ \psi : K \to \mathrm{Aut}(H)$. Prove that $H \rtimes_{\psi} K$ and $H \rtimes_{\psi_2} K$ are isomorphic groups. (Hint: Try the map $H \rtimes_{\psi} K \to H \rtimes_{\psi_2} K$ given by $(h, k) \mapsto (\theta(h), k)$.)

(b) Suppose that $\rho : K \to K$ is an automorphism of K and define $\psi_2 = \psi \circ \rho : K \to \mathrm{Aut}(H)$. Prove that $H \rtimes_{\psi} K$ and $H \rtimes_{\psi_2} K$ are isomorphic groups.

2. Suppose that p and q are primes with $p < q$ where p divides $q - 1$. Show that there are precisely two groups of order pq up to isomorphism.

3. Classify groups of order 20 (there are 5 isomorphism types). Give a presentation of each of the nonabelian groups you find. You do not need to carefully prove that the presentations are correct.

4. Show that every nonabelian group of order $(3)(11)^2$ is isomorphic to a semidirect product $(\mathbb{Z}_{11} \times \mathbb{Z}_{11}) \rtimes_{\phi} \mathbb{Z}_3$, and that there is only one such nonabelian group up to isomorphism.

5. Let G be finite. Show that G has a unique largest solvable normal subgroup H. (in other words, H is solvable and normal and contains every other solvable and normal subgroup of G).

6. Recall that a proper subgroup M of a group G is *maximal* if there does not exist a subgroup H of G with $M \subsetneq H \subsetneq G$. Suppose that G is finite and has the property that every maximal subgroup of G has prime index. Prove that G is solvable, in the following steps.

(a) Prove that if P is a Sylow p-subgroup of G and $N_G(P) \leq H \leq G$ for some subgroup H, then $|G : H| \equiv 1 (\mathrm{mod}\, p)$. (This part is true in any finite group.) (Hint: P is a Sylow p-subgroup of H and $N_G(P) = N_H(P)$.)

(b) Taking p to be the largest prime dividing the order of the group G, show that G has a normal Sylow p-subgroup.

(c) Conclude the proof by induction on the order.

7. A group G is called *supersolvable* if G has a normal series

$$G_0 = \{1\} \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

such that for all $1 \leq i \leq n$, G_i/G_{i-1} is a cyclic group (finite or infinite). (Recall that being a normal series means that $G_i \trianglelefteq G$ for all i.)

(a) Show that if G is supersolvable, so is every subgroup and factor group of G.

(b) Show that a finite nilpotent group is supersolvable.

(c) Give an example of a solvable group which is not supersolvable.