Math 200a Fall 2021 Homework 7

Due Friday 11/19/2021 in class

1. Let G be a finite group.

(a) Prove that G is nilpotent if and only if whenever $a, b \in G$ are elements with relatively prime orders, then a and b commute.

(b) Prove that the dihedral group D_{2n} is nilpotent if and only if n is a power of 2.

2. Let P be a finite p-group for a prime p. The *Frattini subgroup* of a group G, denoted $\Phi(G)$, is the intersection of all of the maximal subgroups of G. Recall also that an *elementary abelian p-group* is a group which is isomorphic to a finite direct product of copies of $\mathbb{Z}_p$.

(a) Show that $P/\Phi(P)$ is an elementary abelian p-group. (Hint: Consider the natural map from P to the product of all of the P/M as M ranges over maximal subgroups of P.)

(b) Show that if N is any normal subgroup of P such that P/N is elementary abelian, then $\Phi(P) \subseteq N$. Thus $\Phi(P)$ is the uniquely smallest normal subgroup with the property that factoring it out gives an elementary abelian p-group.

3. Let R be a commutative ring, and consider the ring $R[[x]]$ of formal power series in one variable.

(a) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in the ring $R[[x]]$ if and only if a_0 is a unit in R.

(b) Suppose that $R = F$ is a field. Show that every nonzero ideal of $F[[x]]$ is equal to the principal ideal (x^n) for some $n \geq 0$. Conclude that the only prime ideals of $F[[x]]$ are 0 and (x), and that (x) is the unique maximal ideal of $F[[x]]$.

4. Prove that a ring D is a division ring if and only if the only left ideals of D are 0 and D.

5. Let R be a ring, and consider the matrix ring $M_n(R)$ for some $n \geq 1$. Given an ideal I of R, let $M_n(I)$ be the set of matrices (a_{ij}) such that $a_{ij} \in I$ for all i, j.

Show that every ideal of $M_n(R)$ is of the form $M_n(I)$ for some ideal I of R. Conclude that if R is a division ring, then $M_n(R)$ is a *simple ring*, that is, that $\{0\}$ and $M_n(R)$ are the only ideals of $M_n(R)$. Show however that $M_n(R)$ is not itself a division ring when $n \geq 2$.

6. Let R be a commutative ring, and let $I = (r_1, \ldots, r_n)$ be a nonzero finitely generated ideal of R. Prove that there is an ideal J of R which is maximal among ideals which do not contain I.