Math 200a Fall 2021 Homework 8

Due Friday 12/3/2021 in class

1. Let R be a commutative ring. Prove that if every prime ideal of R is finitely generated, then all ideals of R are finitely generated, in the following steps:

(a) Suppose that R has an ideal which is not finitely generated. Show that there is an ideal P which is maximal under inclusion among the set of non-finitely generated ideals.

(b) Prove that P is prime: Suppose that $xy \in P$, but $x \notin P$ and $y \notin P$. Define $I = P + (x)$ and note that I is finitely generated, say $I = (p_1 + xq_1, \dots, p_n + xq_n)$, where $p_i \in P, q_i \in R$. Let $K = (p_1, \dots p_n)$ and let $J = \{r \in R | rx \in P\}$. Show that $Jx + K = P$, and that therefore P is finitely generated, a contradiction.

2. Let R be a commutative ring.

(a). The ring R is called *local* if it has a unique maximal ideal M. Show that an ideal M is the unique maximal ideal of R if and only if every element of $R - M = \{r \in R | r \notin M\}$ is a unit.

(b) Let R be an integral domain and let P be a prime ideal of R. Let $X = R - P$ be the set of elements in R which are not in P. Consider the localization RX^{-1}. Show that RX^{-1} is a local ring, with unique maximal ideal $PX^{-1} = \{r/x \,|\, r \in P, x \in X\}$.

(c) Note that R/P is a domain, since P is prime. Show that RX^{-1}/PX^{-1} is isomorphic to the field of fractions of R/P.

3. Let R be an integral domain. Let X be a multiplicative system in R not containing 0, and let $D = RX^{-1}$. Show that if R is a Euclidean domain, so is D.

4. Recall that when D is a squarefree integer, then the *ring of integers* in the field $\mathbb{Q}(\sqrt{D}) = \{x + y\sqrt{D} | x, y \in \mathbb{Q}\}$ is the subring $\mathcal{O} = \{a + b\omega | a, b \in \mathbb{Z}\}$ of $\mathbb{Q}(\sqrt{D})$, where $\omega = \sqrt{D}$ if D is congruent to 2 or 3 modulo 4, while $\omega = (1 + \sqrt{D})/2$ if D is congruent to 1 modulo 4. The field $\mathbb{Q}(\sqrt{D})$ has the norm $N(a + b\sqrt{D}) = a^2 - Db^2$, which is multiplicative, i.e. $N(z_1 z_2) = N(z_1)N(z_2)$ for $z_1, z_2 \in \mathbb{Q}(\sqrt{D})$.

(a) Consider the ring of integers $\mathcal{O}$ in $\mathbb{Q}(\sqrt{D})$. Suppose that for every $z \in \mathbb{Q}(\sqrt{D})$, there exists an element $y \in \mathcal{O}$ such that $|N(z - y)| < 1$. Prove that $\mathcal{O}$ is a Euclidean domain with respect to the function $d : \mathcal{O} \to \mathbb{N}$ given by $d(x) = |N(x)|$. (Hint: follow the method of proof we used to show that $\mathbb{Z}[i]$ is a Euclidean domain).

(b) Show that the ring of integers $\mathcal{O}$ is a Euclidean domain when $D = -2, 2, -3, -7$, or -11. (In each case show that part (a) applies).

5. Consider the ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$, in other words the ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$.

(a) Consider $6 = 2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$. Show that all four of the elements $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are irreducible but not prime, and that R is not a UFD.

(b) Consider the ideals $I_2 = (2, 1 + \sqrt{-5})$, $I_3 = (3, 1 - \sqrt{-5})$, $I_3' = (3, 1 + \sqrt{-5})$. Show that $R/I_2 \cong \mathbb{Z}_2$, and $R/I_3 \cong R/I_3' \cong \mathbb{Z}_3$. Conclude that all three ideals are maximal ideals.

(c) Show that $R/(3) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ as rings. (Hint: Chinese Remainder theorem).

(d) Is $R/(2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$?

6. This problem continues the investigations of the ring R in the previous problem.

(a) Prove that I_2, I_3, I_3' are all nonprincipal ideals of R.

(b) Prove that $(I_2)^2 = (2)$, $I_2 I_3 = (1 - \sqrt{-5})$, $I_2 I_3' = (1 + \sqrt{-5})$, and $I_3 I_3' = (3)$. In particular, this gives multiple examples showing that a product of nonprincipal ideals can be principal. Conclude that if the prinicpal ideals in the equation $(2)(3) = (1+\sqrt{-5})(1-\sqrt{-5})$ are expressed as products of maximal ideals, one gets the same result on both sides of the equation up to rearrangement of the ideals. (This is an illustration of the fact that R is a *Dedekind domain*, a type of ring more general than a UFD in which ideals have unique factorizations as products of maximal ideals).