MATH 200A FALL 2021 MIDTERM SOLUTIONS

1 (15 pts).

(a) If P is a Sylow p-subgroup of a finite group G, prove that $N_G(N_G(P)) = N_G(P)$. (This was a homework exercise— I want you to reprove it).

Solution. Note that $P \trianglelefteq N_G(P)$ by definition. Also, since $|P|$ is the maximum power of p dividing $|G|$, $|P|$ is also the maximum power of p dividing $|N_G(P)|$. So P is a Sylow p-subgroup of $N_G(P)$. Since $P \trianglelefteq N_G(P)$, we know that this means that P is the unique Sylow p-subgroup of $N_G(P)$, and in particular $P \operatorname{char} N_G(P)$. Now $N_G(P) \trianglelefteq N_G(N_G(P))$, again by definition. Since

$$P \operatorname{char} N_G(P) \trianglelefteq N_G(N_G(P)),$$

by a result from class we have $P \trianglelefteq N_G(N_G(P))$. Since $N_G(P)$ is the unique largest subgroup of G inside of which P is normal, $N_G(N_G(P)) \subseteq N_G(P)$. The reverse inclusion holds by definition, so $N_G(P) = N_G(N_G(P))$.

(b) Let p be an odd prime. Prove that a group G of order $4p$ has a normal Sylow p-subgroup if and only if G has a subgroup of order $2p$.

Solution. Suppose first that P is a normal Sylow p-subgroup of G. By Cauchy's theorem, G has an element of order 2, say x, and so $H = \langle x \rangle$ is a subgroup of order 2. Now $|P \cap H|$ divides both 2 and p by Lagrange, and so is 1. It follows that $|HP| = |H||P|/|H \cap P| = 2p$. Moreover, since $P \trianglelefteq G$, HP is a subgroup of G.

Conversely, suppose that G has a subgroup K with $|K| = 2p$. We use the same idea as in part (a). Let P be a Sylow p-subgroup of K. Since $|K : P| = 2$, $P \trianglelefteq K$ since index 2 subgroups are always normal. But then $P \operatorname{char} K$. Similarly, $|G : K| = 2$ and so $K \trianglelefteq G$. Since $P \operatorname{char} K \trianglelefteq G$, we get $P \trianglelefteq G$.

(c) Show that A_4 has no subgroup of order 6.

Solution. We use (b). Since $|A_4| = 12 = 4p$ with $p = 3$, if A_4 has a subgroup of order 6, then A_4 has a normal Sylow 3-subgroup by (b). Then A_4 has a unique Sylow 3-subgroup and so has only 2 elements of order 3, the non-identity elements in this subgroup. But A_4 has 8 elements of order 3 (the 3-cycles). This contradiction shows that A_4 cannot have such a subgroup of order 6.

2 (15 pts).

(a) Consider the group with presentation $(x, y | x^2 = 1, y^2 = 1, (xy)^n = 1)$ for some $n \geq 3$. Show that this group is isomorphic to the dihedral group D_{2n}.

Solution. We think of D_{2n} as the set $\{a^ib^j|0 \leq i \leq n-1, 0 \leq j \leq 1\}$ with $2n$ elements and the mulitplication rules $a^n = b^2 = 1, ba = a^{-1}b$.

Let $H = (x, y|x^2 = 1, y^2 = 1, (xy)^n = 1)$. By the universal property of the free group, there is a unique homomorphism $\widetilde{\phi} : F(x, y) \to D_{2n}$ such that $\widetilde{\phi}(x) = ab$, $\widetilde{\phi}(y) = b$. (Note that the elements x and y have order 2, so they must correspond to order 2 elements of D_{2n}— so it is natural to send them to two different reflections in D_{2n}).

We check that x^2, y^2, and $(xy)^n$ are all in the kernel of $\widetilde{\phi}$, since $\widetilde{\phi}(x^2) = abab = aa^{-1}b^2 = 1$, $\widetilde{\phi}(y^2) = b^2 = 1$, and $\widetilde{\phi}((xy)^n) = (ab^2)^n = a^n = 1$. This means from the universal property of a presentation that there is a unique homomorphism $\phi : H \to D_{2n}$ such that $\phi(x) = ab$ and $\phi(y) = b$.

Let us show a number of different methods for completing the proof, all of which appeared on some students' papers.

Method 1:

We showed in class and in the notes that $(a, b|a^n = 1, b^2 = 1, ba = a^{-1}b)$ is actually a presentation of D_{2n}. So we can use this to easily define a homomorphism $\psi : D_{2n} \to H$ as well: Define $\widetilde{\psi} : F(a, b) \to H$ where $\widetilde{\psi}(a) = xy$, $\widetilde{\psi}(b) = y$; check that all of the relations get sent to 1 by $\widetilde{\psi}$ and so there is an induced homomorphism $\psi : D_{2n} \to H$ with $\psi(a) = xy$ and $\psi(b) = y$. But now $\phi\psi(a) = \phi(xy) = abb = a$ and $\phi\psi(b) = b$. So $\phi\psi$ is trivial on a set of generators of D_{2n}, so $\phi\psi = 1_{D_{2n}}$. Similarly one sees that $\psi\phi = 1_H$. So $\psi = \phi^{-1}$ and ϕ is an isomorphism.

Method 2:

Note that D_{2n} is generated by ab and b, since any subgroup containing these elements also contains $abb^{-1} = a$, and therefore contains all elements a^ib^j. Thus ϕ is surjective. We just need to prove that ϕ is injective. Since $x^2 = 1$ in H, $x^{-1} = x$. Similarly, $y^{-1} = y$. Thus any word in the generators x, y is equal in H to a word with positive powers of x and y. Also, since $x^2 = y^2 = 1$, we never need any power greater than 1. It follows that every word in x, y is equal to one of the form $(xy)^i$, $(xy)^ix$, $(yx)^i$, or $(yx)^iy$ for some $i \geq 0$. In other words these are the words with alternating x's and y's. Now $(yx) = y^{-1}x^{-1} = (xy)^{-1} = (xy)^{n-1}$ so the last two forms are unnecessary. Since $(xy)^n = 1$, every word is equal in H to one of the form $(xy)^i$ or $(xy)^ix$ where $0 \leq i < n$. There are $2n$ such words, so $|H| \leq 2n$. Since $|D_{2n}| = 2n$ and ϕ is surjective, ϕ must also be injective and hence an isomorphism.

Method 3:

This is the same as Method 2 but with a different way of conceptualizing how to prove injectivity. We know that in D_{2n} the elements a and b have nice relations between them, so in H we focus on the corresponding elements $z = xy$ and y. Since $zy = (xy)y = x$, and H is generated by x and y, we see that z and y also generate H. Moreover, $z^n = (xy)^n = 1$ and $y^2 = 1$ are relations, and we also have $yz = yxy = (yx)y = (xy)^{-1}y = z^{-1}y$ since

$(yx) = y^{-1}x^{-1} = (xy)^{-1}$. Now every element in H is equal to a word in the generators y and z and in any such word the relation $yz = z^{-1}y$ allows one to move all y's to the right, leaving a word z^iy^j. Since $z^n = 1$ and $y^2 = 1$, every element of G is equal to one of the elements $\{z^iy^j|0 \leq i \leq n-1, 0 \leq j \leq 1\}$, so $|G| \leq 2n$. Otherwise the proof is the same as in Method 2.

(b) Suppose that G is any finite group which has order at least 6 and is generated by two elements of order 2. Show that $G \cong D_{2n}$ for some $n \geq 3$.

Solution.

Let G be generated by elements b and c of order 2. The element (bc) has finite order since G is finite, say $|bc| = n$. Since $b^2 = c^2 = 1$ and $(bc)^n = 1$, there is a homomorphism $\phi : H = (x, y|x^2 = y^2 = (xy)^n = 1) \to G$ with $\phi(x) = b$ and $\phi(y) = c$ and since b and c generate G, ϕ is surjective. We know that $H \cong D_{2n}$, so $|H| = 2n$. So $|G| \leq 2n$. Since G has an element of order n, $|G|$ is a multiple of n. If $|G| = n$, this means that G is generated by bc, so G is cyclic of order n. But then G can only have at most one element of order 2, and cannot be generated by elements of order 2 unless $|G| = 2$, contradicting the hypothesis. Thus $|G| > n$, which forces $|G| = 2n$. Now ϕ has to be an isomorphism, so $G \cong H \cong D_{2n}$.