Math 200b 1/4/2021

Modules.

Conventions on rings - unital, not commutative

<u>Def.</u> R a ring.
A left R-module is an abelian group $(M, +)$ with an R-action
$$R \times M \longrightarrow M$$
$$(r, m) \longmapsto r \cdot m.$$
s.t.

1. $r \cdot (s \cdot m) = (rs) \cdot m$
2. $1 \cdot m = m.$
3. $r \cdot (m+n) = r \cdot m + r \cdot n$
4. $(r+s) \cdot m = r \cdot m + s \cdot m$

$\forall r, s \in R, \ \forall m, n \in M.$

Rmks -

- these axioms force $0 \cdot m = 0$, $(-1) \cdot m = -m$.
- We usually write rm instead of $r \cdot m$.

Examples

① $R = F$ a field. Then an F-module is just a vector space.

② Any ring R, $M = R$ is a left R-module where

$r \cdot s = rs$.

③ F a field, $V = F^n$ column vectors of length n

$$V = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \; a_i \in F \right\}$$

$R = M_n(F)$ acts on V by matrix multiplication

$A \in R$, $v \in V$,

$A \cdot v = Ax$

(4) $R = \mathbb{Z}$. A $\mathbb{Z}$-module is just an abelian group, M. Recall if $n \in \mathbb{Z}$ then we define for $m \in M$

$$n \cdot m = \begin{cases} \overbrace{m + \cdots + m}^{n} & n \geq 1 \\ 0 & n = 0 \\ \overbrace{(-m) + \cdots + (-m)}^{|n|} & n \leq -1 \end{cases}$$

and this makes M into a $\mathbb{Z}$-module.

(5) Let $\phi : R \to S$ be a ring homomorphism. Let M be an S-module. Then M is also an R-module where $r \cdot m = \phi(r) \cdot m$.
"restriction of scalars"

Def. Let M, N be R-modules. A homomorphism $f: M \to N$ is a homomorphism of abelian groups ($f(m_1 + m_2) = f(m_1) + f(m_2)$) s.t. $f(rm) = rf(m)$. $\leftarrow$
$\forall\, r \in R$, $m, m_1, m_2 \in M$

Ex. If $R = F$ is a field, an F-module homomorphism $f: V \to W$ is a linear transformation over F.

Ex. Let R be a left R-module by left multiplication.
For any $x \in R$, $f_x : R \to R$
$r \mapsto rx$

$f_x =$ "right mult. by x"
is a module homomorphism since

$$f_x(r \cdot s) = rsx = r \cdot f_x(s)$$

but "left multiplication by x" is just an abelian group map, not an R-module homomorphism (unless R is commutative)

<u>Ex.</u> if $R = \mathbb{Z}$ then a homomorphism of $\mathbb{Z}$-modules is just an homomorphism of abelian groups.

<u>Def.</u> M an R-module. A subset $N \subseteq M$ is a <u>submodule</u> if N is a subgroup under $+$, and $rx \in N$ $\forall r \in R$ $x \in N$.

<u>Ex.</u> for any R-module M, $\{0\}$ and M are submodules of M.

<u>Ex.</u> If $f: M \to N$ is an R-module homomorphism, then

$$\operatorname{Ker} f = \{m \in M \mid f(m) = 0\}$$

and $\operatorname{Im} f = f(M)$

are submodules of M and N respectively.

Ex. Let R be a module by left mult. A submodule of R is a left ideal I, since $rx \in I$ for $r \in R$, $x \in I$.

Ex. If V is an F-module, F a field, an F-submodule of V is just a vector subspace.

Ex. If F^n is a left $M_n(F)$-module, then 0 and F^n are the only submodules. this is a simple module.

Def. Let M be a module over R and let N be a submodule. We define the factor module M/N to be the abelian group $M/N = \{m+N \mid m \in M\}$ with R-action $r\cdot(m+N) = rm+N$.

Ex. If I is a left ideal of a ring R, we have a factor module R/I which is a left R-module, where $r\cdot(s+I) = rs+I$.

1st $\cong$ theorem:

__Thm.__ Let $f: M \to N$ is an R-module homomorphism, then there is an $\cong$ of R-modules

$$\bar{f}: M/(\ker f) \longrightarrow f(N).$$
$$m + (\ker f) \longrightarrow f(m).$$

__Pf.__ By group thm, $\bar{f}$ is an $\cong$ of Abelian groups. Then

$$\bar{f}(r \cdot (m + \ker f))$$
$$= \bar{f}(rm + \ker f)$$
$$= f(rm)$$
$$= r f(m)$$
$$= r \cdot \bar{f}(m + \ker f).$$

so $\bar{f}$ is a module $\cong$.

Module structures on Hom.

__Def.__ Let M, N be R-modules.

$\mathrm{Hom}_R(M, N) = \{ f: M \to N \mid f \text{ is an } R\text{-module homomorphism} \}$

This is a set, but it actually has more structure.

- $\mathrm{Hom}_R(M, N)$ is always an Abelian group. If $f, g \in \mathrm{Hom}_R(M, N)$ define $f+g \in \mathrm{Hom}_R(M, N)$, where $[f+g](m) = f(m) + g(m)$ "pointwise sum".
  This is a group with $0 = 0$-function and $[-f](m) = -f(m)$

- Next time - sometimes an R-module (R commutative)
  sometimes a ring (when $M = N$)