Lec. 10 1/27/2021

Tensor products.

Motivation. Let F be a field. V an F-space. $\dim_F V = \infty$.

Define $V^* = \mathrm{Hom}_F(V, F)$

"linear functionals on V"

"dual space of V".

There is a natural pairing

$$\theta : V^* \times V \longrightarrow F$$
$$(\varphi, v) \longmapsto \varphi(v)$$

but θ is not linear

(considering $V^* \times V$ as $V^* \oplus V$).

instead –

$$\theta(\psi + \phi, v) = (\psi + \phi)(v) = \psi(v) + \phi(v)$$
$$= \theta(\psi, v) + \theta(\phi, v)$$

$$\Theta(\phi, v+w) = \phi(v+w) = \phi(v) + \phi(w)$$
$$= \Theta(\phi, v) + \Theta(\phi, w)$$

We say Θ is <u>bilinear</u> - linear separately in each coordinate.

But $\Theta : V^* \oplus V \longrightarrow F$
$$(\varphi, v) \longrightarrow \varphi(v)$$

is not F-linear since

$$\Theta((\phi, v) + (\psi, w)) = \Theta(\phi + \psi, v + w)$$
$$= \Theta(\phi, v) + \Theta(\psi, w) + \Theta(\psi, v) + \Theta(\phi, w)$$
$$\neq \Theta(\phi, v) + \Theta(\psi, w).$$

Want a way to replace Θ by a linear map with the same information.

We will define a tensor product $V^* \otimes V$ and a linear map
$$\tilde{\Theta} : V^* \otimes V \longrightarrow F$$
with the same information as Θ.

Def. Let R be a ring.
Let M be a right R-module
N a left R-module.
P an Abelian group.

A function $\phi: M \times N \to P$ is **R-balanced** if

- $\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$
- $\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$.
- $\phi(mr, n) = \phi(m, rn)$

$\forall r \in R, \ \forall m, m_1, m_2 \in M$
$\forall n, n_1, n_2 \in N$.

Ex. M a left R-module.
R right R-module by multiplication.

$$\varphi : R \times M \longrightarrow M.$$
$$(r, m) \longmapsto r \cdot m$$

is R-balanced.

$$\varphi(rs, m) = rs \cdot m$$
$$\|$$
$$\varphi(r, sm) = r \cdot s \cdot m$$

Def. M a right R-module, N left R-mod.
A tensor product of M and N over R is an abelian group T and a R-balanced map $\theta : M \times N \longrightarrow T$

$$(m, n) \longmapsto \theta(m, n)$$

s.t. for any R-balanced map
$$\varphi : M \times N \longrightarrow P$$
there is a unique homomorphism of abelian groups $\psi : T \longrightarrow P$ s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & T \arrow[d, dashed, "\psi"] \\ & P \end{tikzcd}$$

commutes.

$(\psi \circ \theta = \varphi)$.

Note ψ has all of the information in φ since
$$\varphi = \psi \circ \theta.$$

Ex. If M is a left R-module
$$R \times M \longrightarrow M \qquad (r, m) \longmapsto r \cdot m$$
is a tensor product of R and M over R.

Thm. Tensor products are unique up to $\cong$.

M right R-module, N left R-module,
$\theta_1 : M \times N \longrightarrow T_1$ $\quad \theta_2 : M \times N \longrightarrow T_2$
both tensor products. Then there is a unique isomorphism of abelian groups $\varphi : T_1 \longrightarrow T_2$ s.t. $\varphi \circ \theta_1 = \theta_2$.

Pf. — same as other universal property proofs of uniqueness.

Thm. If M is a right R-module, N left R-module. Then there exists a tensor product $\theta : M \times N \longrightarrow T$.

Pf.

Consider $S = M \times N$

write $(m, n) \in S$ as $m \otimes n$

F will be a free $\mathbb{Z}$-module with basis $\{ \underline{m \otimes n} \mid (m, n) \in S \}$.

So an element of F looks like

$a_1 (m_1 \otimes n_1) + \cdots + a_k (m_k \otimes n_k)$

some $a_i \in \mathbb{Z}$, $m_i \in M$, $n_i \in N$

Let $T = F / \underline{I}$ where

$\underline{I}$ is the subgroup generated by all elements of the form

- $(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n$
- $m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2$
- $mr \otimes n - m \otimes rn$

$\forall\ m, m_1, m_2 \in M,\ n, n_1, n_2 \in N,\ r \in R.$

Let $\theta : M \times N \longrightarrow T = F/I$
$(m, n) \longmapsto m \otimes n + I$

So θ is R-balanced.

Also θ is a tensor product if $\phi : M \times N \longrightarrow P$ is R-balanced. We need

$$\begin{tikzcd} M \times N \arrow[r, "\theta"] \arrow[dr, "\phi"'] & T \arrow[d, dotted, "\psi"] \\ & P \end{tikzcd}$$

$\psi : T = F/I \longrightarrow P$ a homomorphism s.t. $\psi \circ \theta = \phi$.

There is a unique $\mathbb{Z}$-linear map $\hat{\psi} : F \longrightarrow P$
$\underline{m \otimes n} \longrightarrow \underline{\phi(m, n)}$

(F is free)

Since ϕ is R-balanced, check $I \subseteq \ker \hat{\varphi}$.

(all of the generators of I are in $\ker \hat{\varphi}$, so $I \subseteq \ker \hat{\varphi}$)

So $\hat{\varphi}$ induces $\psi : F/I \longrightarrow P$

$$m \otimes n + I \longrightarrow \phi(m,n).$$

and $\psi \circ \theta = \phi$.

Also ψ is unique (check)

Rmk.

Given a tensor product

$\theta : M \times N \longrightarrow T$

we write T as $M \otimes_R N$

we call $M \otimes_R N$ the tensor product

and don't write θ in the notation.

But we write $\theta(m, n)$ as $m \otimes n \in T$.

Careful: an arbitrary element of T

is an element of the form

$$\sum_{i=1}^{d} a_i (m_i \otimes n_i) \qquad a_i \in \mathbb{Z}, \; m_i \in M, \; n_i \in N$$

by the proof.

$$= \sum_{i=1}^{d} (a_i m_i \otimes n_i) \qquad \text{(linearity in the first coordinate)}$$

$$= \sum_{i=1}^{d} (m_i' \otimes n_i) \qquad m_i' \in M$$

(*) an element of $M \otimes_R N$ looks like

$$\sum_{i=1}^{d} (m_i \otimes n_i) \quad \text{where } m_i \otimes n_i$$

are <u>pure</u> <u>tensors</u>.