Lecture 11 1/29/2021

Last time:

for any right R-module M, left module N, we defined $M \otimes_R N$ an abelian group with

$$\otimes : M \times N \longrightarrow M \otimes_R N$$
$$(m,n) \longmapsto m \otimes n$$

pure tensor

s.t. $\otimes$ is R-balanced and if $\phi: M \times N$ is R-balanced then $\exists$ unique Ψ $\mathbb{Z}$-module map s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "\phi"'] & M \otimes_R N \arrow[d, dotted, "\Psi"] \\ & P \end{tikzcd}$$

Note $\otimes$ satisfies

$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$

$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$

$mr \otimes n = m \otimes rn.$

Ex. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = ?$

A general element looks like

$$\sum (a_i + n\mathbb{Z}) \otimes \frac{p_i}{q_i} \quad a_i \in \mathbb{Z},\ p_i, q_i \in \mathbb{Z}$$

$q_i \neq 0.$

$$= \sum (a_i + n\mathbb{Z}) \otimes \frac{(n)p_i}{nq_i}$$

$$= \sum (a_i + n\mathbb{Z})n \otimes \frac{p_i}{nq_i}$$

$$\sum (na_i + n\mathbb{Z}) \otimes \frac{p_i}{nq_i}$$

$$= \sum (0 + n\mathbb{Z}) \otimes \frac{p_i}{nq_i} = 0.$$

So $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$

(note $0 \otimes n = 0$ in $M \otimes_R N$

since $0 \otimes n = (0+0) \otimes n = 0 \otimes n + 0 \otimes n$

$\Rightarrow 0 = 0 \otimes n.$

Similarly $m \otimes 0 = 0.$)

When is $M \otimes_R N$ a module?

<u>Lemma.</u> if $f: M \to M'$ is a homomorphism of right modules, then there is a $\mathbb{Z}$-module homomorphism (for any left module N)

$$f \otimes 1 : M \otimes_R N \longrightarrow M' \otimes_R N$$
$$(m \otimes n) \longmapsto (f(m) \otimes n).$$

<u>Pf.</u> Define

$$\widehat{f \otimes 1} : M \times N \longrightarrow M' \otimes_R N$$
$$(m, n) \longmapsto f(m) \otimes n$$

check that it is R-balanced.

e.g. $(mr, n) \longmapsto f(mr) \otimes n$

$$= f(m)r \otimes n$$
$$= f(m) \otimes rn$$
$$= \widehat{f \otimes 1}(m, rn)$$

univ. property says $\exists$ unique homomorphism of groups

$$f\otimes 1 : M\otimes_R N \longrightarrow M'\otimes_R N$$

s.t. $\underline{m\otimes n} \longmapsto f(m)\otimes n.$

$$\begin{tikzcd}
(m,n) \arrow[ddd, bend right, mapsto] & m\otimes n \arrow[ddd, bend left, mapsto] \\
M\times N \arrow[r, "\otimes"] \arrow[dr, "\widehat{f\otimes 1}"'] & M\otimes N \arrow[d, dashed, "f\otimes 1"] \\
& M'\otimes_R N \\
& f(m)\otimes n
\end{tikzcd}$$

The general formula for $f\otimes 1$ is

$$\sum_{i=1}^{d} m_i\otimes n_i \mapsto \sum f(m_i)\otimes n$$

Lemma ① If M is an (S,R)-bimodule then
$M \otimes_R N$ is a left S-module,
where $s\cdot(m\otimes n) = (sm\otimes n)$

② if N is an (R,T)-bimodule
$M\otimes_R N$ is a right T-module
$(m\otimes n)\cdot t = (m\otimes nt)$.

Pf. By previous lemma

$l_s : M \longrightarrow M$ is a

right R-homomorphism

since $l_s(mr) = \underline{s(mr)}$

$= \underline{(sm)r} = l_s(m)r.$

$l_s \otimes 1 : M \otimes N \longrightarrow M \otimes N$

$$(m \otimes n) \longmapsto (sm \otimes n)$$

is a homomorphism of groups.

It makes sense to define

$$s \cdot (m \otimes n) = sm \otimes n$$

to give $M \otimes N$ a left S-action.

Then easy to check module axioms.

② Similar.

<u>Ex.</u> M a left module.

I ideal of R.

R/I is an (R,R)-bimodule.

So $R/I \otimes_R M$

is a left-R-module

And $\cong \dfrac{M}{IM}$

as a left R-module.

$$\phi : R/I \times M \longrightarrow \frac{M}{IM}$$

$$(\underline{r+I}, m) \longmapsto \underline{r}m + \underline{I}M$$

- well defined
- R-balanced

$\exists$

$$\psi : R/I \otimes_R M \longrightarrow \frac{M}{IM}$$

$$(r+I \otimes m) \longmapsto rm + IM$$

a group homomorphism.

Actually, a homomorphism

of left R-modules.

$\varphi(s \cdot (r + I \otimes m))$

$= \varphi(sr + I \otimes m)$

$= srm$

$= s \cdot \varphi(r + I \otimes m)$

Now check φ is <u>iso</u>.

The commutative case.

Let R be commutative.

A right R-module M is a left R-module. $r \cdot m = mr$

So we define $M \otimes_R N$ for left modules M, N.

Then $M \otimes_R N$ is always an R-module again because M is an (R, R)-bimodule and

$$\underline{r} \cdot (m \otimes n) = (r \underline{\underline{m}} \otimes n)$$

$$= (mr \otimes n) = (m \otimes rn)$$

Universal property can be stated differently:

Def. if M, N, P are R-modules (R commutative)

$\phi: M \times N \longrightarrow P$

is **R-bilinear** if

$$\phi(r_1 m_1 + r_2 m_2, n)$$

$$= r_1 \phi(m_1, n) + r_2 \phi(m_2, n)$$

$$\phi(m, r_1 n_1 + r_2 n_2)$$

$$= r_1 \phi(m, n_1) + r_2 \phi(m, n_2).$$

Then $\otimes : M \times N \longrightarrow M \otimes_R N$

is R-bilinear and

if $\Phi: M \times N \longrightarrow P$

is R-bilinear there is

a unique R-module

homomorphism $\psi: M \otimes_R N \to P$

s.t. $\Phi = \psi \circ \theta$

$$\begin{tikzcd} M \times N \arrow[r, "\theta"] \arrow[dr, "\phi"'] & M \otimes_R N \arrow[d, dashed, "\psi"] \\ & P \end{tikzcd}$$

Rmk. if M is a left R-module it is a right R-module $m * r = rm$ and an (R,R)-bimodule

$$s(m*r) \qquad (sm)*r$$
$$s(rm) \quad = \quad r(sm)$$

Case of fields.

Thm. let F a field. V, W vector spaces over F
$\{v_i \mid i \in I\}$ basis for V
$\{w_j \mid j \in J\}$ " " W.

Then $V \otimes_F W$ is a vector space over F with basis $\{ v_i \otimes w_j \mid i \in I, j \in J \}$

So $\dim_F (V \otimes_F W) = (\dim_F V)(\dim_F W)$

<u>Pf.</u> A general element of $V \otimes_F W$ is $\sum_{k=1}^{m} t_k \otimes u_k$

$$t_k = \sum_{i \in I} a_{ik} v_i$$

$$u_k = \sum_{j \in J} b_{jk} w_j$$

$$t_k \otimes u_k = \sum_{i \in I} \sum_{j \in J} a_{ik} b_{jk} (v_i \otimes w_j)$$

So $\{(v_i \otimes w_j) \mid i \in I, j \in J\}$

spans.

independence — see notes.