Lecture 12 21/1/2021

– Extension of scalars.

$\phi: R \longrightarrow S$ a ring homomorphism.

We saw: if M is a left S-module, M is naturally a left R-module

where $r * m = \phi(r) m$ $r \in R$.

"restriction of scalars".

If M is a left R-module, we can form a left S-module:

<u>Def.</u> $\phi: R \to S$ a ring homom.

Given a left R-module M, the extension of scalars is the left S-module $\underline{S \otimes_R M}$.

where $s \cdot (\underline{t \otimes m}) = \underline{st \otimes m}$.

Rmk. S is an (S,R)-bimodule where it is a right R-module using ϕ: $s * r = s\phi(r)$

So $S \otimes_R M$ is a left S-module.

Ex. If $F \subseteq K$ is an inclusion of fields. Then if V is a v.s. over F, $K \otimes_F V$ is a v.s. over K.

- if $\{v_i\}_{i\in I}$ is a F-basis of V, $\{1\otimes v_i\}_{i\in I}$ is a K-basis of $K\otimes_F V$.

So $\dim_F V = \dim_K K\otimes_F V$.

if $\beta = \{v_1, \dots v_n\}$ is a basis of V, (so $\beta' = \{1\otimes v_1, \dots, 1\otimes v_n\}$ is a K-basis of $K\otimes_F V$,)

If $\psi: V \to U$ is F-linear,

then $M^{\beta}_{\beta}(\varphi) = M^{\beta'}_{\beta'}(1\otimes\varphi)$

$$1\otimes\varphi : K\otimes_F V \longrightarrow K\otimes_F V$$
$$(a\otimes v) \longmapsto (a\otimes\varphi(v))$$

<u>Ex.</u> $\mathbb{R}\subseteq\mathbb{C}$ $V=\mathbb{R}^2$

Then $\mathbb{C}\otimes_{\mathbb{R}} V = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^2$

$$\mathbb{C}\otimes_{\mathbb{R}}(\mathbb{R}\oplus\mathbb{R}) = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}\oplus\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}$$

$$\underline{\mathbb{C}}\oplus\underline{\mathbb{C}}.$$

<u>Ex.</u> $\mathbb{Q}\otimes_{\mathbb{Z}} \underline{\mathbb{Z}/n\mathbb{Z}} = 0$

as we've seen.

$\mathbb{Z}/n\mathbb{Z}$ with scalars extended from $\mathbb{Z}$ to $\mathbb{Q}$ $(i:\mathbb{Z}\to\mathbb{Q})$

is 0.

Algebras.

Def. Let R be a commutative ring. A *R-algebra* is a ring A which is also a module over R s.t. for all $r \in R$, $a, b \in A$

$$r \cdot (ab) = (r \cdot a) b = a (r \cdot b).$$

Ex. R commutative,

$R[x]$ is an R-algebra.

$M_n(R)$ " "

$R[[x]]$ " "

if $R \subseteq S$ S commutative,

S is an R-algebra.

<u>Rmk.</u> if A is an R-algebra then
$$\phi : R \longrightarrow A$$
$$r \longmapsto (r \cdot 1)$$
is a homomorphism of rings.

for $\phi(rs) = (rs)\cdot 1 = r\cdot(s\cdot 1)$
$= r\cdot(1(s\cdot 1)) = (r\cdot 1)(s\cdot 1)$

Also $\phi(R) \subseteq Z(A)$ the center of A.

$(r\cdot 1)b = r\cdot(1b) = r\cdot b = r\cdot(b1)$
$= b(r\cdot 1) \quad \forall\, b \in A.$

Conversely: if A is a ring and $\phi : R \longrightarrow A$ is a homomorphism with $\phi(R) \subseteq Z(A)$ then A is an R-algebra where $r\cdot a = \phi(r)a$.

Thm. Let A, B be R-algebras over a commutative ring R. Then $A \otimes_R B$ is an R-algebra, where

$$(a \otimes b)(c \otimes d) = (ac \otimes bd).$$

Pf. (omitted)

Ex. R commutative ring, S any R-algebra. Then $S \otimes_R R[x]$ is an R-algebra and it is isomorphic to $S[x]$.

Ex. if S is an R-algebra,

$$S \otimes_R M_n(R) \cong M_n(S)$$

as R-algebras (even S-algebras)

idea of proof:

check: $S \times M_n(R) \longrightarrow M_n(S)$

$(s, A) \longmapsto sA$

is R-bilinear.

So you get

$\psi: S \otimes_R M_n(R) \longrightarrow M_n(S)$

$(s \otimes A) \longmapsto sA.$

(and ψ is an algebra map)

Also

$\theta: M_n(S) \longrightarrow S \otimes_R M_n(R)$

$\sum a_{ij} e_{ij} \longmapsto \sum_{i,j} a_{ij} \otimes e_{ij}$

$$e_{ij} =: \begin{array}{c} \quad j \\ \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \end{array}$$

check Θ, Ψ are inverses.

$$\Psi \circ \Theta \left(\sum a_{ij} e_{ij} \right)$$

$$\Psi \left(\sum_{i,j} a_{ij} \otimes e_{ij} \right)$$

$$= \sum_{i,j} a_{ij} e_{ij} \qquad \Psi \circ \Theta = \underline{1}.$$

<u>Ex.</u> $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

is an $\mathbb{R}$-algebra.

and $\dim_{\mathbb{R}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = 4$.

this is isomorphic to $\mathbb{C} \underline{\times} \mathbb{C}$.

So $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is <u>not</u> a domain why?

$$\big((1\otimes i)-(i\otimes 1)\big)\big((1\otimes i)+(i\otimes 1)\big) = 0.$$

$$(1\otimes i^2)-(i\otimes i)+(i\otimes i)-(i^2\otimes 1)$$

$$= -(1\otimes 1)+(1\otimes 1) = 0.$$

Note

$\{(1\otimes 1), (1\otimes i), (i\otimes 1), (i\otimes i)\}$

is an $\mathbb{R}$-basis.

__Def.__ A sequence of maps of left R-modules

$$M \xrightarrow{f} N \xrightarrow{g} P.$$

is exact at N if

$$\operatorname{Im} f = f(M) = \ker g.$$

A short exact sequence is

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

which is exact at M, N and P.

So $\ker f = 0$ i.e. f is injective, $f(M) = \ker g$, and $g(N) = P$.

So by 1st $\cong$ thm,

$$\frac{N}{f(M)} \cong P.$$

$$f(M) \cong M.$$

N is called an <u>extension</u> of P by M.

Q? if $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is short exact, and Q is a right R-module, is

$$0 \to Q \otimes_R M \xrightarrow{1 \otimes f} Q \otimes_R N \xrightarrow{1 \otimes g} Q \otimes_R P \to 0$$

exact?

Ans. not always.