Lecture 13 2/3/2021.

Thm.

Let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be a short exact sequence of left R-modules. Let Q be a right R-module. Then

$$Q \otimes_R M \xrightarrow{1 \otimes f} Q \otimes_R N \xrightarrow{1 \otimes g} Q \otimes_R P \to 0$$

is exact.

here

Recall: this means $1 \otimes g$ is surjective, $\operatorname{im}(1 \otimes g) = \ker 0 = Q \otimes_R P$ and

$$\operatorname{Im}(1 \otimes f) = \ker(1 \otimes g)$$

In general, $1 \otimes f$ is not injective.

Pf. Let $q \otimes p \in Q \otimes_R P$.

Since g is surjective, there is $n \in N$ s.t. $g(n) = p$.

Then $[1 \otimes g](q \otimes n) = q \otimes p$.

So $\mathrm{Im}(1 \otimes g)$ contains all pure tensors, so $= Q \otimes_R P$.

Next, since $g \circ f = 0$ so

$(1 \otimes g) \circ (1 \otimes f) = 0$

So $\mathrm{Im}(1 \otimes f) \subseteq \mathrm{Ker}(1 \otimes g)$.

Define $L = Q \otimes_R N / \mathrm{Im}(1 \otimes f)$

which is a left R-module.

We have a map

$$1\otimes g : \frac{Q\otimes_R N}{\mathrm{Im}(1\otimes f)} \longrightarrow Q\otimes_R P$$

$$q\otimes n + \mathrm{Im}(1\otimes f) \longrightarrow q\otimes g(n).$$

Define

$$\psi : Q\otimes_R P \longrightarrow \frac{Q\otimes_R N}{\mathrm{Im}(1\otimes f)}$$

$$q\otimes p \longmapsto q\otimes n + \mathrm{Im}(1\otimes f)$$

where $n\in N$ is any element s.t. $g(n)=p$.

- Check independent of choice of n.

if $g(n')=p=g(n)$ then $g(n-n')=0$

So $n-n'\in \ker g = \mathrm{Im}(f)$. Say

$f(m)=n-n'$ Then

$$q\otimes n - q\otimes n' = q\otimes(n-n') = q\otimes f(m)$$

$\in \mathrm{Im}(1\otimes f)$ so

$$q\otimes n + \mathrm{Im}(1\otimes f) = q\otimes n' + \mathrm{Im}(1\otimes f)$$

- use universal property to define ψ.

Claim: $\varphi_0(\overline{1\otimes g}) = 1_L$

$\varphi_0(\overline{1\otimes g})(q\otimes n + \mathrm{Im}(1\otimes f))$

$\varphi(q\otimes g(\underline{n}))$

$= q\otimes n + \mathrm{Im}(1\otimes f)$.

So $\overline{1\otimes g}$ is injective

$$\mathrm{Ker}\,\overline{1\otimes g} = \frac{\mathrm{Ker}(1\otimes g)}{\mathrm{Im}(1\otimes f)} = 0$$

So $\mathrm{Ker}(1\otimes g) = \mathrm{Im}(1\otimes f)$.

Ex. (failure of left exactness of $\otimes$)

Let $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$ $\quad n \geq 2$.

$a \longmapsto an$

which is a homomorphism of left $\mathbb{Z}$-modules.

f is injective, so fits into a short exact sequence $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/n\mathbb{Z} \to 0$

Apply $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} -$

$$\begin{tikzcd}
\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "1 \otimes f"] \arrow[d, "\cong"'] & \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z}/n\mathbb{Z} \arrow[r, "\hat{f}"] & \mathbb{Z}/n\mathbb{Z}
\end{tikzcd}$$

$(\bar{a} \otimes 1)$

(recall: $M \otimes_R R = M$ in fact $M \otimes_R R/I = M/IM$)

$\bar{a} \longmapsto n\bar{a} = \bar{0}$.

So $1 \otimes f = 0$ so it is not injective

<u>Def.</u> A right R-module Q is <u>flat</u> (over R) if for all

short exact seqs of left R-modules $0 \to M \to N \to P \to 0$

$$0 \to Q \otimes_R M \to Q \otimes_R N \to Q \otimes_R P \to 0$$

is short exact.

Ex. $\mathbb{Z}/n\mathbb{Z}$ $n \geq 2$ is not a flat $\mathbb{Z}$-module.

Rmk. Q is flat iff for any injective homomorphism $f: M \to N$, $1 \otimes f : Q \otimes_R M \to Q \otimes_R N$ is injective.

Lemma. Let $\{M_\alpha\}_{\alpha \in I}$ be a family of right R-modules. Let N be a left R-module. Then

$$\left(\bigoplus_{\alpha \in I} M_\alpha\right) \otimes_R N \cong \bigoplus_{\alpha \in I} (M_\alpha \otimes_R N)$$

(as Abelian groups, or as R-modules, if R is commutative)

Pf. (omit)

Rmk. The corresponding result for products is false.

Thm. If F is a free right module then F is flat.

Pf: Let $f: M \longrightarrow N$ be an injective left R-module homomorphism. We need $1 \otimes f : F \otimes_R M \longrightarrow F \otimes_R N$ to be injective still.

Write $F \cong \bigoplus_{\alpha \in I} R$.

$$\begin{tikzcd}
F \otimes_R M \arrow[r, "1\otimes f"] \arrow[d, equal] & F \otimes_R N \arrow[d, equal] \\
\left(\bigoplus_{\alpha \in I} R\right) \otimes_R M \arrow[r, "1\otimes f"] \arrow[d, "\cong"] & \left(\bigoplus_{\alpha \in I} R\right) \otimes_R N \arrow[d, "\cong"] \\
\bigoplus_{\alpha \in I} (R \otimes_R M) \arrow[r] \arrow[d, "\cong"] & \bigoplus_{\alpha \in I} (R \otimes_R N) \arrow[d, "\cong"] \\
\bigoplus_{\alpha \in I} M \arrow[r, "h"] & \bigoplus_{\alpha \in I} N
\end{tikzcd}$$

Check

$$h((m_\alpha)) = (f(m_\alpha))$$

Since f is injective, so is h.

Ex. Let R be commutative and X a multiplicative system in R. Then RX^{-1} is a flat R-module. (Pf in 200C)

Ex. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Def. Let P be a left R-module. P is *projective* if given any surjective map of left R-modules $M \xrightarrow{g} N$ and a homomorphism $f: P \to N$ then there exists $h: P \to M$ s.t.

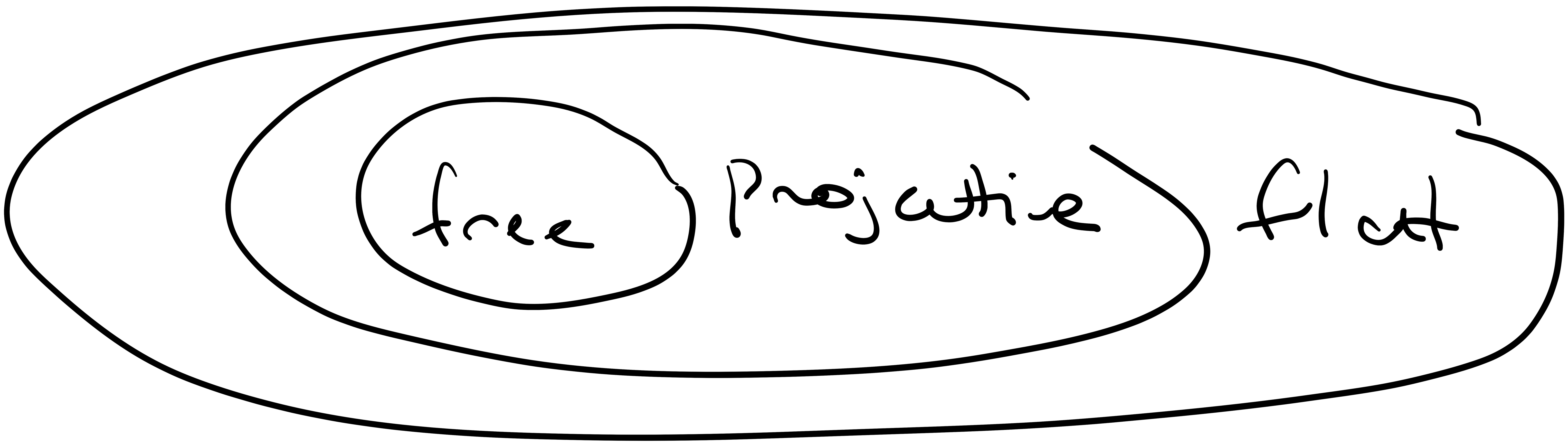
$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dotted, "\exists h"'] & \\ M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

$g \circ h = f.$

<u>Ex.</u> Free left R-modules are projective. We used it to show surjections onto free modules are split.

<u>Thm.</u> A module P is projective iff there is a module Q s.t. $P \oplus Q = F$ is free.

<u>Cor.</u> Projective modules are flat.



Ex. Let $R = M_2(F)$

F a field. Then $R = I \oplus J$

for left ideals

$$I = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$$

So I and J are projective.

But I can't be free, since

$\bigoplus_{i=1}^{n} R$ has dimension $4n$

over F, while $\dim_F I = 2$.