Lec 16 2/12/21

- Algebraic Extensions

Def. Let $F \subseteq K$ be a field extension, it is algebraic if for all $\alpha \in K$, α is algebraic over F.

Recall $\alpha \in K$ is alg over F if $\exists\, 0 \neq f \in F[x]$ s.t. $f(\alpha) = 0$. Then there is a minimal poly $\text{minpoly}_F(\alpha)$ = unique monic irreducible $f \in F[x]$ s.t. $f(\alpha) = 0$.
$\to$ also monic poly of min. possible degree s.t. $f(\alpha) = 0$.

Lemma. Let $E \subseteq F \subseteq K$ be fields. Then $[K:E] = [K:F][F:E]$.

Pf. This means if $[K:F] = \infty$ or $[F:E] = \infty$ then $[K:E] = \infty$. (check)

Assume now $[K:F] < \infty$ and $[F:E] < \infty$.
Let $\{\alpha_1, \ldots, \alpha_m\}$ be an E-basis of F

Let $\{\beta_1, \dots, \beta_n\}$ be an F-basis of K.

We prove $\{\alpha_i \beta_j \mid 1 \le i \le m, 1 \le j \le n\} = S$ is a E-basis of K.

If $\gamma \in K$, then $\gamma = \sum_{i=1}^{n} a_i \beta_i \quad \underline{a_i \in F}$

$= \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{m} (b_{ij}\alpha_j)\, \beta_i} \quad b_{ij} \in E.$

So S spans K over E.

If $\sum_{i,j} b_{ij} \alpha_j \beta_i = 0 \quad b_{ij} \in E.$

$\sum_i \left(\sum_j b_{ij} \alpha_j\right) \beta_i = 0 \quad \left(\sum_j b_{ij} \alpha_j\right) \in F$

$\Rightarrow \sum_j b_{ij} \alpha_j = 0 \quad \forall i \quad (\beta_i \text{ ind. over } F)$

$\Rightarrow b_{ij} = 0 \quad \forall i \, \forall j \quad (\alpha_j \text{ ind. over } E).$

So S is ind. over E, so a basis of K over E.

Cor. If $E \subseteq F \subseteq K$ with $[K:E]$ finite, then $[E:F]$ and $[K:F]$ divide $[K:E]$.

Ex. if $[K:E] = p$ is prime and $E \subseteq F \subseteq K$ then $F = E$ or $F = K$.

Cor. If $F \subseteq K$ is an extension with $[K:F] < \infty$, then it is algebraic.

Pf. If $\alpha \in K$, then $F \subseteq F(\alpha) \subseteq K$

So $[F(\alpha) : F] < \infty$.

Then α is algebraic over F.

Prop. Let $F \subseteq K$. Let $\alpha, \beta \in K$ be algebraic over F. Then $\alpha \pm \beta$, $\alpha\beta$, α^{-1} (if $\alpha \neq 0$) are also algebraic over F.

Pf. We have $[F(\alpha) : F] < \infty$ and $[F(\beta) : F] < \infty$.

$$[F(\underline{\alpha, \beta}) : F] = [F(\alpha)(\beta) : F(\alpha)]\underbrace{[F(\alpha) : F]}_{< \infty}$$

if $0 \neq f \in F[x]$ s.t. $f(\beta) = 0$

then $f \in F(\alpha)[x]$ so β is alg. over $F(\alpha)$, so $[F(\alpha)(\beta) : F(\alpha)] < \infty$.

(in fact it is $\leq [F(\beta):F]$

So $[F(\alpha,\beta):F] < \infty$.

Then $\alpha \pm \beta$, $\alpha\beta$, $\alpha^{-1} \in F(\alpha,\beta)$.

So they are all algebraic over F.

Ex. $\sqrt{2}+\sqrt{3}$ is alg. over $\mathbb{Q}$.

$\text{minpoly}_{\mathbb{Q}}(\sqrt{2}) = x^2-2$,

$\text{minpoly}_{\mathbb{Q}}(\sqrt{3}) = x^2-3$.

How do we find $f \in \mathbb{Q}[x]$ s.t. $f(\sqrt{2}+\sqrt{3})=0$

$$\alpha^2 = (\sqrt{2}+\sqrt{3})^2 = 2+2\sqrt{6}+3$$

$$2\sqrt{6} = \alpha^2 - 5.$$

$$24 = (\alpha^2-5)^2$$

$f(\alpha)=0$ where $f = (x^2-5)^2 - 24$

$$= x^4 - 10x + 1.$$

Def. $F \subseteq K$ is f.g. (as a field extension) if $K = F(\alpha_1, \dots, \alpha_n)$ some $\alpha_i \in K$.

Lemma. Let $F \subseteq K$. Then:

$[K:F] < \infty$ iff $F \subseteq K$ is algebraic and finitely generated.

Pf. if $[K:F] < \infty$, we saw it is algebraic, and f.g. (for example by a basis for K over F)

Conversely, if $K = F(\alpha_1, \dots, \alpha_n)$ and each α_i is alg. over F. Then

$$[F(\alpha_1, \dots, \alpha_{i+1}) : F(\alpha_1, \dots, \alpha_i)]$$
$$\leq [F(\alpha_{i+1}) : F] := e_{i+1}$$

Since if $f \in F[x]$ s.t. $f(\alpha_{i+1}) = 0$ then $f \in F(\alpha_1, \dots, \alpha_i)[x]$.

Thus $\text{minpoly}_{F(\alpha_1, \dots, \alpha_i)}(\alpha_{i+1})$

$$\leq \text{minpoly}_F(\alpha_{i+1})$$

Then

$$[F(\alpha_1, \dots, \alpha_n) : F] =$$

$$[F(\alpha_1, \dots, \alpha_n) : F(\alpha_1, \dots, \alpha_{n-1})][F(\alpha_1, \dots, \alpha_{n-1}) : F(\alpha_1, \dots, \alpha_{n-2})]$$

$$\dots \quad [F(\alpha_1) : F]$$

$\leq e_n e_{n-1} \cdots e_1 < \infty.$

So $[K:F] < \infty$.

<u>Cor.</u> If $K = F(\alpha_1, \ldots, \alpha_n)$ and $e_i = \deg \operatorname{minpoly}_F(\alpha_i) < \infty$.
Then $[K:F] \leq e_1 e_2 \cdots e_n$.

<u>Ex.</u> $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}] = 6$

it is ≤ 6 by the corollary since
$\operatorname{minpoly}_{\mathbb{Q}}(\sqrt[3]{2}) = x^3 - 2$
$\operatorname{minpoly}_{\mathbb{Q}}(\sqrt{3}) = x^2 - 3$.

Also $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}]$ is divisible by $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$.

<u>Thm.</u> Let $E \subseteq F \subseteq K$ where $E \subseteq F$ and $F \subseteq K$ are algebraic.
Then $E \subseteq K$ is algebraic.

<u>Pf.</u> Let $\alpha \in K$. Then α is alg.

over F, say

$\text{minpoly}_F(\alpha) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$\in F[x]$.

Consider $E(a_0, \ldots, a_{n-1}) \subseteq F$

Now α is algebraic over $E(a_0, \ldots, a_{n-1})$.

So $[E(a_0, \ldots, a_{n-1})(\alpha) : E(a_0, \ldots, a_{n-1})]$

$< \infty$.

Also $[E(a_0, \ldots, a_{n-1}) : E] < \infty$.

since each a_i is alg. over E.

Now $[E(a_0, \ldots, a_{n-1}, \alpha) : E] < \infty$

So α is alg. over E.

So K is alg. over E.

Lemma: Let $E \subseteq K$ be a field extension.

Let $F = \{\alpha \in K \mid \alpha \text{ is alg. over } E\}$.

Then (i) F is a field.

③ if $F \subseteq L \subseteq K$ and L/F is algebraic then $L = F$.
"F is alg. closed in K".

Pf. ① Since if $\alpha, \beta \in F$ then $\alpha \pm \beta$, $\alpha\beta$, $\alpha^{-1} \in F$.

② Since F/E is alg. so if L/F is alg then L/E is alg. Then $L \subseteq F$, so $L = F$.

Ex. $\mathbb{Q} \subseteq \mathbb{C}$. Then

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is alg. over } \mathbb{Q}\}$$

is the algebraic closure of $\mathbb{Q}$.