Lecture 2 1/6/21

- Recall - M, N R-modules

$\mathrm{Hom}_R(M,N) = \{ f : M \to N \mid f$ is a homomorphism of R-modules. $\}$

This is Abelian group, $f, g \in \mathrm{Hom}_R(M,N)$

$f+g \in \mathrm{Hom}_R(M,N)$ with

$[f+g](m) = f(m) + g(m)$.

- Suppose R is commutative.

Then $\mathrm{Hom}_R(M,N)$ is an R-module with

$r \in R$, $f \in \mathrm{Hom}_R(M,N)$

$[rf](m) = f(rm) = rf(m)$

Note $(r \cdot (s \cdot f))(m) = [s \cdot f](rm)$

$= f(srm)$

while

$[(rs) \cdot f](m) = f(rsm)$

$=$ since R is commutative.

- Suppose $M = N$. Then

$\mathrm{Hom}_R(M,M) = \mathrm{End}_R(M)$

is the ring of endomorphisms of M.

It is a ring with $f, g \in \mathrm{End}_R(M)$

$$fg = f \circ g.$$

ring axioms are routine, e.g.

$$(f+g)h = fh + gh \quad f, g, h \in \mathrm{End}(M)$$

$$\begin{aligned}
[(f+g) \circ h](m) &= [f+g](h(m)) \\
&= f(h(m)) + g(h(m)) \\
&= [f \circ h](m) + [g \circ h](m) \\
&= [f \circ h + g \circ h](m) \\
&= [fh + gh](m).
\end{aligned}$$

identity element $= 1_M : M \to M$
$m \mapsto m$.

Ex. F a field, V a F-module (vector space) $\dim V = n < \infty$

Then $\mathrm{End}_F(V) = \{$l.t. $f : V \to V\}$

fix a basis. Then $\mathcal{B} = \{v_1, \ldots, v_n\}$.

$$\mathrm{End}_F(V) \cong M_n(F).$$

$\phi \longmapsto (a_{ij})$ where

$$\phi(v_j) = \sum_i a_{ij} v_i$$

Ex. Let R be a left R-module by left multiplication.

$$\mathrm{End}_R(R) \cong R^{op}$$

where $R^{op} = R$ but multiplication $r * s = sr$.

$$\phi : \mathrm{End}_R(R) \longrightarrow R^{op}$$
$$f \longmapsto f(1)$$

ϕ homomorphism of rings –

$\phi(f+g) = (f+g)(1) = f(1) + g(1)$
$= \phi(f) + \phi(g)$.

$\phi(fg) = f(g(1))$,

Notice $r \in R$, $f(r) = f(r1)$
$= r f(1)$.

So $f(g(1)) = g(1) f(1)$

$\phi(f) * \phi(g) = \phi(g)\phi(f)$
$= g(1) f(1)$ ✓

bijective:

if $r \in R^{op}$ then

$\phi_r \in End(R)$ where $\phi_r(x) = xr$

satisfies $\phi_r(1) = r$.

so surjective

Injectivity: if $\phi(f) = 0$ then

$f(1) = 0$ so $f(r) = rf(1) =$

$r0 = 0$ so $f = 0$.

Another point of view on modules.

– if a group G acts on X,

then we can think of it as

a homomorphism $G \to Sym(X)$.

$$g \mapsto [x \mapsto g \cdot x].$$

This generalizes to modules:

<u>Thm.</u> There is a bijection between

(for fixed ring R, abelian group M)

$$\left\{ \begin{array}{l} R\text{-module structures} \\ \text{on } M \end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{l} \text{ring homomorphisms} \\ \theta : R \to \mathrm{End}_{\mathbb{Z}}(M) \end{array} \right\}$$

where $\Phi(M) = \theta$ where
$$[\theta(r)](m) = r \cdot m.$$

Pf (sketch)

Given a module M, why is θ a ring homomorphism?

First $\theta(r) \in \mathrm{End}_{\mathbb{Z}}(M)$ since

$$\theta(r)[m_1 + m_2] =$$
$$r\cdot(m_1+m_2) = r\cdot m_1 + r\cdot m_2$$
$$= [\theta(r)](m_1) + [\theta(r)](m_2)$$

(module axiom ③)

Then θ is a ring homomorphism.

e.g. $[\theta(rs)](m) = (rs)\cdot m$
$$= r\cdot(s\cdot m) = r\cdot[\theta(s)](m)$$
$$= \theta(r)(\theta(s)(m))$$
$$= [\theta(r) \circ \theta(s)](m)$$

$$\theta(rs) = \theta(r) \circ \theta(s)$$

(used module axioms (1))

Rest left to you.

Application: $F[x]$ - modules.

Let V be a $F[x]$ - module.

Then since $F \subseteq F[x]$, V is an F-module (i.e. a F-vector space).

Also we have
$$\phi : V \longrightarrow V$$
$$v \longmapsto x \cdot v$$

Then if $a \in F$, $xa = ax$

So $xa \cdot v = ax \cdot v$

So $x \cdot (av) \quad\quad a \cdot (x \cdot v)$

$\phi(av) = a\,\phi(v)$.

Then $\phi \in \mathrm{End}_F(V)$

i.e. ϕ is a linear transformation.

Thm. F a field. Fix a Abelian group V.

There is a bijection between

$$\left\{ \begin{array}{l} F[x]\text{-module} \\ \text{structures on } V \end{array} \right\} \text{ and }$$

$$\downarrow \ \underline{\overline{\Phi}}$$

$$\left\{ \begin{array}{l} F \text{ vector space structures on } V \\ \text{together with a linear trans} \\ \qquad \phi : V \to V. \end{array} \right\}$$

where

$\underline{\overline{\Phi}}(V) = V$ as a F-module via restricted action and $\phi(v) = x \cdot v$.

Pf. (Sketch).

Given an F vector space V and $\varphi \in \mathrm{End}_F(V)$,

Define a homomorphism

$$\theta : F[x] \longrightarrow \mathrm{End}_{\mathbb{Z}}(V)$$

by $\sum a_i x^i \longmapsto \left[v \mapsto \sum a_i \varphi^i(v) \right]$

$(\varphi^0 = \mathrm{id}_V)$

which corresponds to an $F[x]$-module structure on V for which

$$x \cdot v = \varphi(v).$$