Lec 25 3/10/2021

<u>Thm</u> (Galois)

Let F be a field of char 0.
Let $f \in F[x]$ is solvable by radicals
iff $\mathrm{Gal}(K/F)$ is <u>solvable</u>
where K is the splitting field of f
over F.

<u>Ex.</u> Let $f = 2x^5 - 10x + 5 \in \mathbb{Q}[x]$.

Then f is irreducible by Eisenstein.
with prime 5. Let K = splitting field of
f over $\mathbb{Q}$. We're going to show $\underbrace{\mathrm{Gal}(K/\mathbb{Q})}_{G}$
$\cong S_5$, which is not solvable
So f is not solvable by radicals.

If $\alpha \in K$ is a root of f, $[\mathbb{Q}(\alpha):\mathbb{Q}] = 5$.
since $f = \mathrm{minpoly}_{\mathbb{Q}}(\alpha)$.

Note also if $\sigma \in G$, σ permutes the roots of f, $\{\alpha_1, \dots, \alpha_5\}$, and is determined by that permutation, since $K = \mathbb{Q}(\alpha_1, \dots, \alpha_5)$

So $G \subseteq S_5$.

$5 \mid |G|$ since $[K:\mathbb{Q}]$ is a mult. of 5.

So G has an element of order 5 (Cauchy) which must be a 5-cycle.

We claim f has 3 real roots.

Note $f' = (2x^5 - 10x + 5)'$

$= 10x^4 - 10$

real roots of f' are ± 1

So graph of f looks like 

and $f(-1) = 13$ $f(1) = -3$.

Calculus. $\alpha_1, \alpha_2, \alpha_3$ are real, say the other 2 roots α_4, α_5 are complex and $\alpha_5 = \overline{\alpha_4}$ since $f \in \mathbb{R}[x]$.

If $\tau \in \mathrm{Aut}(\mathbb{C})$ is complex conjugation,

$\tau(\alpha_i) = \alpha_i$ $1 \le i \le 3$

$\tau(\alpha_4) = \alpha_5$ $\tau(\alpha_5) = \tau(\alpha_4)$.

τ restricts to an aut of K.
and $\tau|_K \in G = \langle (45) \rangle$

Now check — any subgroup of S_5 containing a 5-cycle and a 2-cycle is all of S_5.

So $G = S_5$.

Alg. closed. fields.

<u>Def.</u> A field K is alg. closed. if for all nonconstant $f \in K[x]$, then f has a root in K.

<u>Rmk.</u> if K is alg. closed then for all nonconstant f, f actually splits in $K[x]$ as $f = c(x-\alpha_1)\cdots(x-\alpha_n)$.
(factor thm + induction.)

<u>Def.</u> If F is a field, $F \subseteq K$ a field extension, K is an <u>algebraic closure</u> of F if K/F is algebraic

and K is alg. closed.

<u>Lemma.</u> Let $F \subseteq K$ be an algebraic extension. TFAE.

1. K is alg. closed. (i.e. an alg. closure of F)
2. if $K \subseteq L$ is algebraic, then $L = K$.
3. If $f \in F[x]$, then f splits in $K[x]$.

<u>Pf.</u> Assume 3. Suppose $K \subseteq L$ is alg. Then $F \subseteq L$ is algebraic. If $\alpha \in L$ then $\mathrm{minpoly}_F(\alpha) \in F[x]$ splits in $K[x]$. So $\alpha \in K$, so 2 holds.

If 2 holds, given $f \in K[x]$ let $K \subseteq L$ be a splitting field for f over K. Then L/K is algebraic, so $K = L$. So f splits over K already, so 1.

1 $\Rightarrow$ 3 is obvious.

Thm. If F is a field, F has an algebraic closure K.

Pf.

1. It's enough to find $F \subseteq L$ where L is alg. closed. If we have that, take
$K = \{\alpha \in L \mid \alpha \text{ is alg. over } F\} \subseteq L$.
Then K/F is alg. And if $f \in K[x]$ then f splits over L, so the splitting field of f over K inside L, say E
$F \subseteq \overbrace{K \subseteq E}$ $\subseteq L$ has E/K algebraic, so E/F is alg, so $K = E$, so f splits over K, and K is alg. closed.
So $F \subseteq K$ is an alg. closure.

2. We need to find an alg. closed L.
(trick due to Emil Artin)
For every nonconstant poly $f \in F[x]$, define a variable x_f and a ring $R = F[x_f \mid f \in F[x]]$

Let I be the ideal generated by all polys $\{f(x_f) \mid f \in F[x]\}$.

$I \neq R$: if not, $1 \in I$ so

$$1 = \sum_{i=1}^{n} g_i f_i(x_{f_i}) \quad g_i \in R.$$

for some distinct $f_1, \dots, f_n \in F[x]$.

Take K to the splitting field over F of $f_1 f_2 \cdots f_n$. So each f_i has a root in K, say α_i

Define a homomorphism

$$\begin{aligned} \phi : R = F[x_f \mid f \in F[x]] &\longrightarrow K \\ a \in F &\longrightarrow a. \\ x_{f_i} &\longmapsto \alpha_i \\ x_g &\longmapsto 0 \\ g \neq f_i & \end{aligned}$$

Then

$$1 = \phi(1) = \sum_{i=1}^{n} \phi(g_i) f_i(\alpha_i) = 0.$$

a contradiction. So $I \neq R$.

3. Choose a maximal ideal M of R with $I \subseteq M \subsetneq R$. (Zorn)

Then $L_1 = R/M$ is a field

$F \hookrightarrow F[x_f] = R \longrightarrow R/M = L_1$

is injective (F a field) so we can think of $F \subseteq L_1$

Now if $f \in F[x]$ is nonconstant, then $x_f + M \in R/M = L_1$ is a root of f since $f(x_f + M) =$

$f(x_f) + M$ but $f(x_f) \in I \subseteq M$

so $f(x_f) + M = 0 + M = 0$.

So every nonconstant $f \in F[x]$ has a root in L_1.

Finally, similarly there is $L_1 \subseteq L_2$ s.t. every $f \in L_1[x]$ has a root in L_2.

Define $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$ inductively.

Let $L = \bigcup_{i \geq 1} L_i$. Then L is alg. closed:

If $g \in L[x]$, then every coefficient of g lies in some L_i, so $g \in L_n[x]$

Then g has a root in $L_{n+1} \subseteq L$

So every nonconstant $g \in L(x)$ has a root in L

So L is alg. closed.