Lecture 3 1/8/21

Direct products + sums.

__Def:__ Let $\{M_\alpha \mid \alpha \in I\}$ is a collection of R-modules.

Then the direct __product__ is

$\prod_{\alpha \in I} M_\alpha$ = the cartesian product

with R-action $r \cdot (m_\alpha) = (r m_\alpha)$

and Abelian group structure given by the usual direct product $(m_\alpha) + (n_\alpha) = (m_\alpha + n_\alpha)$.

The direct __sum__ is

$$\bigoplus_{\alpha \in I} M_\alpha = \left\{ (m_\alpha) \in \prod_\alpha \;\middle|\; \begin{array}{l} m_\alpha = 0 \text{ for all but} \\ \text{finitely many } \alpha \end{array} \right\}$$

which is an R-submodule of $\prod M_\alpha$.

If I is finite then

$M_1 \times \dots \times M_n = M_1 \oplus \dots \oplus M_n$.

Generation of Modules.

<u>Def.</u> If M is an R-module, $X \subseteq M$ a subset, the submodule of M generated by X = the smallest submodule containing X
(= the intersection of all submodules containing X)

Explicitly this is
$$RX = \{ r_1x_1 + \cdots + r_nx_n \mid r_i \in R,\ x_i \in X \}.$$

M is f.g. (finitely generated) if there is some finite $X \subseteq M$ that generates M.
if M is generated by $\{x\}$ we say M is <u>cyclic</u>. ($M = Rx$).

<u>Ex.</u> If $R = \mathbb{Z}$ a cyclic $\mathbb{Z}$-module is a cyclic Abelian group.

<u>Ex.</u> Let R be a commutative ring. A cyclic submodule of R is Rx for some $x \in R$, i.e. a principal ideal.

Ex. When F is a field, if V is an F-module and $X \subseteq V$ then the submodule generated by X is the span of X.

Ex. Let R be arbitrary, M a cyclic left R-module. Then $M \cong R/I$ where I is a left ideal of R.

If $M = Rm$, i.e. $\{m\}$ generates M, define
$$f : R \longrightarrow M \qquad r \longmapsto rm.$$
which is an R-module homomorphism. Then $f(R) = M$. $\ker(f)$ is some R-submodule, so left ideal I of R. The 1st $\cong$ thm says
$$R/I \cong M \text{ as modules.}$$
Conversely R/I is always cyclic, gen. by $\{1+I\}$.

Free modules.

Def. A R-module F is free on a subset $X \subseteq F$ if given any function $f : X \to M$ where M is an R-module there is a unique R-module homomorphism $g: F \to M$ s.t. $g|_X = f$.

Picture
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\exists!\, g"] \\ & M \end{tikzcd}$$
i: inclusion f.n.

$g \circ i = f$

Thm. If F is free on X and G is free on Y then if $|X| = |Y|$ then $F \cong G$ as R-modules.

Pf. (sketch).

There is a bijection $h: X \to Y$.
Then h extends to a homomorphism $f: F \to G$.
and h^{-1} " " " $g: G \to F$
check $f \circ g = 1_G$, $g \circ f = 1_F$.

Thm. Let $F = \bigoplus_{\alpha \in I} R$ for some index set I. Then F is free.

Pf. Define $e_\beta = (r_\alpha)_{\alpha \in I}$ where $r_\beta = 1$ and $r_\alpha = 0$ $\alpha \neq \beta$.
"standard basis vectors"
Claim: F is free on $\underbrace{\{e_\beta \mid \beta \in I\}}_{= X}$.

Given a function $f: X \longrightarrow M$

M a module, define $g: F \longrightarrow M$

by $g\left((r_\alpha)_{\alpha \in I} \right) = \sum_{\alpha \in I} r_\alpha f(e_\alpha)$

Check this is a homomorphism s.t. $g|_X = f$

Also g is unique since

$(r_\alpha)_{\alpha \in I} = \sum_{\alpha \in I} r_\alpha e_\alpha$, so

$$\begin{aligned} g((r_\alpha)) &= \sum_{\alpha \in I} g(r_\alpha e_\alpha) \\ &= \sum_{\alpha \in I} r_\alpha g(e_\alpha) \\ &= \sum_{\alpha \in I} r_\alpha f(e_\alpha) \end{aligned}$$

<u>Cor.</u> Every free R-module is isomorphic to $\bigoplus_{\alpha \in I} R$ for some index set I.

<u>Pf.</u> if F is free on X, then since $\bigoplus_{\alpha \in X} R$ is free on a set of the same cardinality,

$$F \cong \bigoplus_{\alpha \in X} R.$$

Def. Let F be an R-module

Then $X \subseteq F$ is a **basis** if

1) X generates F

2) if $x_1, \dots, x_n$ are distinct elements in X

and $r_1 x_1 + \dots + r_n x_n = 0$

$r_i \in R$, then $r_i = 0$

for all i.

Ex. If K is a field, V is an K-module

a basis for V means what it always means.

Thm. An R-module M is free on $X \subseteq M$ iff X is a basis for M.

Pf. (sketch)

Check if X is a basis then every $m \in M$ is uniquely expressible as

$$m = r_1 x_1 + \dots + r_n x_n$$

with $r_i \in R$, $x_1, \dots, x_n$ distinct elements of X.

Then use this to prove

$$M \cong \bigoplus_{x \in X} R$$

Cor. if K is a field, every K-module is free.
(since every K-module has a basis).

Rmk. There are rings R s.t. $R \cong R \oplus R$ as R-modules.