Lecture 4 1/11/21.

Internal direct sums.

→ if $N_1, \dots, N_m \subseteq M$ are submodules then $N_1 + \dots + N_m =$
$$\{ n_1 + \dots + n_m \mid n_i \in N_i \}.$$
which is an R-submodule, the <u>sum</u> of the submodules.

<u>Thm.</u> M an R-module.
$N_1, \dots, N_m$ R-submodules of M. if

(i) $N_1 + \dots + N_m = M$ and

(ii) $N_i \cap (N_1 + \dots + N_{i-1} + N_{i+1} + \dots + N_m) = 0$ for all i.

Then $M \cong N_1 \oplus \dots \oplus N_m$.

and we say M is an <u>internal</u> <u>direct</u> <u>sum</u> of the N_i.

<u>Pf.</u> By the group version,
$$\phi : N_1 \oplus \dots \oplus N_m \longrightarrow M$$
$$(n_1, \dots, n_m) \longmapsto n_1 + \dots + n_m$$

is an $\cong$ of Abelian groups
Now notice ϕ is a homomorphism of R-modules.

<u>Def.</u> A surjective module homomorphism $f: N \to M$ is <u>split</u> if there is a homomorphism $g: M \to N$ s.t. $f \circ g = 1_M$.

<u>Lemma</u> if $f: N \to M$ is split then $N \cong M \oplus (\ker f)$

<u>Pf.</u> Let $M' = g(M)$. $f \circ g = 1_M$ $\Rightarrow$ g is injective. Then $M' \cong M$
We'll show N is internal direct sum of M' and $(\ker f) = K$.

- $M' \cap K = 0$: if $x \in M' \cap K$, $x = g(y)$ $y \in M$, $f(x) = 0$.

$f(x) = f(g(y)) = y = 0$

So $x = g(y) = g(0) = 0$.

- $M' + K = N$:

If $y \in N$, consider $y - g(f(y))$.

Then $f(y - g(f(y))) =$

$f(y) - \underline{f(g}(f(y)))$

$= f(y) - f(y) = 0$.

So $y - g(f(y)) \in K = \ker f$.

So $y = g(f(y)) + (y - g(f(y)))$

$\in M' + K$.

So $N \cong M' \oplus K$

$\cong M \oplus K$.

<u>Cor.</u> If $f : M \to F$ is a R-module surjection, and if F is <u>free</u> then f is split, so $M \cong F \oplus (\ker f)$.

__Pf.__ We want $g: F \to M$ s.t. $f \circ g = 1_F$. Let $\{e_\alpha\}$ be a basis for F. For each α fix $m_\alpha \in M$ s.t. $f(m_\alpha) = e_\alpha$. Now there is a unique $g: F \to M$ s.t. $g(e_\alpha) = m_\alpha$ and so $f \circ g(e_\alpha) = e_\alpha$ for all α, So $f \circ g = 1_F$. Now apply the lemma to get $M \cong F \oplus (\ker f)$.

Now: Let R be commutative.
Goal: understand f.g. modules over a PID R.

__Def.__ If M is an R-module, $m \in M$
$\operatorname{ann}_R(m) = \{r \in R \mid rm = 0\}$
the annihilator of m.
The annihilator of M is
$\operatorname{ann}_R(M) = \{r \in R \mid rm = 0 \ \forall m \in M\}$

$= \bigcap_{m \in M} \text{ann}_R(m).$

Note $\text{ann}_R(m)$, $\text{ann}_R(M)$ are ideals of R.

<u>Def.</u> Let R be an integral domain, M an R-module.
$m \in M$ is <u>torsion</u> if $\text{ann}_R(m) \neq 0$.
i.e. $rm = 0$ for some $r \neq 0$ in R.
Otherwise m is non-torsion.
$\text{Tors}(M) = \{ m \in M \mid m \text{ is torsion} \}$.
M is <u>torsion</u> if $M = \text{Tors}(M)$
and M is <u>torsionfree</u> if $\text{Tors}(M) = 0$.

<u>Lemma.</u> R an integral domain, M a module.
① $\text{Tors}(M)$ is a submodule of M.
② $M/\text{Tors}(M)$ is torsionfree.

<u>Pf.</u> if $m_1, m_2 \in \text{Tors}(M)$,
$r_1 m_1 = 0$ $r_2 m_2 = 0$ of $r_1, r_2 \in R$
Then $r_1 r_2 (m_1 - m_2) = 0$
and $r_1 r_2 \neq 0$ so $m_1 - m_2 \in \text{Tors}(M)$.
Also $r_1(sm_1) = 0$ $\forall s \in R$, so $sm_1 \in \text{Tors}(M)$.

(2). if $r \neq 0$ and $r(m + \mathrm{Tors}(M)) = 0$

then $rm + \mathrm{Tors}(M) = 0$

So $rm \in \mathrm{Tors}(M)$.

Then $s(rm) = 0$ $s \neq 0$.

Now $sr \neq 0$ so $m \in \mathrm{Tors}(M)$

So $m + \mathrm{Tors}(M) = 0$.

So $M/\mathrm{Tors}(M)$ is torsion free.

<u>Ex.</u> $R = \mathbb{Z}$.

A module M is torsion iff every element of M has finite order. M is torsion free iff all $m \neq 0$ in M has infinite order. e.g. $\mathbb{R}$ is torsion free.

e.g. $\mathbb{Z}/(m)$ is torsion with annihilator $= (m)$.

Modules over PIDs.

<u>Thm.</u> Let R be a PID. Let M be a f.g. R-module. Then

① $M \cong \overbrace{R \oplus \dots \oplus R}^{r} \oplus R/(p_1^{e_1}) \oplus \dots \oplus R/(p_m^{e_m})$

for some primes $p_1, \dots, p_m$. $e_i \geq 1$.

r is the <u>rank</u> of M

$p_1^{e_1}, \dots, p_m^{e_m}$ are the <u>elementary divisors</u>.

② Also, r and the elementary divisors are unique, up to rearranging the elementary divisors or replacing p_i by an associate.

Remarks.

① M being f.g. is essential. $\mathbb{Q}$ is a $\mathbb{Z}$-module which is not a direct sum of cyclic $\mathbb{Z}$-modules. (so not free).

$\mathbb{Q}$ can't be an internal direct sum $N \oplus P$ since if $0 \neq N \subseteq \mathbb{Q}$, $0 \neq P \subseteq \mathbb{Q}$ then

$N \cap P \neq 0$.

② R being a PID is essential, since $Rx + Ry \subseteq k[x, y]$

is not a direct sum of cyclics.