Lecture 5 1/13/21

<u>Thm.</u> M f.g. module over a PID R.
Then

① $M \cong \overbrace{R \oplus \cdots \oplus R}^{r} \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_m^{e_m})$

where p_i are prime, $e_i \geq 1$.

r = <u>rank</u> of M

$p_1^{e_1}, \ldots, p_m^{e_m}$ are <u>elementary divisors</u>.

② M' f.g., $M \cong M'$
iff M and M' have same rank,
same elementary divisors
(up to order and associates)

Torsionfree case.

<u>Prop.</u> Let M be f.g. torsionfree over the PID R. Then M is free of finite rank.

<u>Pf.</u> Let $M = Rm_1 + \cdots + Rm_n$.

Induct on n. If $n=0$, $M=0$ is free of rank 0.

Now assume $n \geq 1$ and the result holds for smaller n.

Consider M/Rm_1 and its torsion submodule $\mathrm{Tors}(M/Rm_1)$

$= K/Rm_1$ where $K = \{m \in M \mid rm \in Rm_1 \text{ some } 0 \neq r\}$

Then $(M/Rm_1)/(K/Rm_1)$ is torsionfree

$\cong M/K$ as modules.

Note M/K is generated by $m_1+K, \dots, m_n+K$ but $m_1+K = 0+K$ so $m_2+K, \dots, m_n+K$ generate M/K. By induction M/K is free of finite rank.

Since $\pi: M \to M/K$ is a surjection onto a free, π is split so $M \cong (M/K) \oplus K$.

Just need K free. In fact K is free of rank 1.

$$Rm_1 \subseteq K = \{m \in M \mid \underset{r \neq 0}{rm \in Rm_1}\}$$

K/Rm_1 is torsion.

Note K is f.g. (a summand of M)

K/Rm_1 is torsion, and f.g.

Then $\mathrm{Ann}_R(K/Rm_1) \neq 0$.

(if $x_1, \dots, x_s$ are generators, and $r_i x_i = 0$ $r_i \neq 0$, $r_1 r_2 \cdots r_s \neq 0$ kills the module)

Pick $x \in \mathrm{Ann}(K/Rm_1)$.

So $xK \subseteq Rm_1$.

Notice $xK \cong K$ (M torsion free)

$Rm_1 \cong R$

So it is enough to show a submodule of R is free. Since R is a PID submodules are xR which are 0 or $\cong R$. $\square$.

Ex. if $R = k[x,y]$, k a field, and $I = xR + yR$, then I is torsionfree, not free.

Check I is not cyclic.

I is not a direct sum of 2 smaller submodules: if $I = J \oplus L$, J, L nonzero ideals, then $0 \neq x \in J$, $0 \neq y \in L$, $0 \neq xy \in J \cap L$, contradicting that the sum is direct.

<u>Cor.</u> if M is a submodule of a f.g. free module F over a PID, then M is free. (actually true for arbitrary free modules over a PID)

Torsion modules.

<u>Def.</u> Let $p \in R$ be prime (R a PID). A module M is

p-primary if

for all $m \in M$, $p^i m = 0$

for some $i \geq 1$.

If M is p-primary

and $m \in M$, $\mathrm{ann}_R(m) = (p^i)$

for some i. If M is f.g.

$\mathrm{ann}_R(M) = (p^i)$ some i

Prop. Let M be f.g. torsion

over a PID R.

Then for some primes p_i,

$M \cong M_{p_1} \oplus \cdots \oplus M_{p_k}$

where M_{p_i} is

$= \{ m \in M \mid (p_i)^j m = 0 \text{ some } j \}$

is a p_i-primary submodule of M.

<u>Pf.</u> M f.g. n torsion means

$\operatorname{ann}_R(M) \neq 0$.

Say $\operatorname{ann}_R(M) = (a)$

$a = p_1^{e_1} \cdots p_k^{e_k}$

in R, p_i are non-associate primes, $e_i \geq 1$.

Each M_{p_i} is a p_i-primary submodule of M.

Use the criteria for internal direct sums to check

$M \cong M_{p_1} \oplus \cdots \oplus M_{p_k}$.

Ex. if $R = \mathbb{Z}$

$\mathbb{Z}/a\mathbb{Z}$,

$a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$

we already knew

$\mathbb{Z}/a\mathbb{Z} \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{e_k}\mathbb{Z}$.

<u>Prop.</u> Let M be a f.g. p-primary R-module. Then

$$M \cong R/(p^{i_1}) \oplus \cdots \oplus R/(p^{i_k})$$

some $i_1, i_2, \ldots, i_k \geq 1$.

Idea.

Suppose $M = R/(p^{i_1}) \oplus \cdots \oplus R/(p^{i_k})$

Look at

pM - factors $\underline{\underline{R/(p)}}$ will die

$$p\left(R/(p^{i_1})\right) = R(p + (p^{i_1}))$$

this killed by $p^{i_1 - 1}$, $\cong$ to

$$R/(p^{i_1 - 1})$$

Also look at

$M[p] = \{ m \in M \mid pm = 0 \}$

$= R(p^{i_1 - 1} + (p^{i_1})) \oplus \cdots \oplus R(p^{i_k - 1} + (p^{i_k}))$

and you can think of this as a vector space over $R/(p)$ which is a field.

Pf (sketch).

Look at pM.

if $\mathrm{ann}_R(M) = (p^n)$

then $\mathrm{ann}_R(pM) = (p^{n-1})$.

Induct on n.

$pM \cong R/(p^{j_1}) \oplus \cdots \oplus R/(p^{j_\ell})$

$= C_1 \oplus \cdots \oplus C_l$

where C_i is cyclic

$C_i = R g_i \quad g_i \in pM$.

Choose $h_i \in M$ s.t.

$p h_i = g_i$ and show

$R h_1 \oplus \cdots \oplus R h_l$

is also direct.

This gives the part of

M except the needed

copies of $R/(p)$.

Look at $M[p]$ to

find those.