Lecture 7 1/20/2021

Canonical forms.

Review.

Let V f.d. vectorspace $/ F$.

Let $\phi : V \longrightarrow V$ a linear trans.

If $\beta = \{v_1, \dots, v_n\}$ is a basis, we define

$$M_\beta^\beta(\phi) = (a_{ij}) \qquad \phi(v_j) = \sum_i a_{ij} v_i$$

Can define $\det(\phi) = \det(a_{ij})$

If $\beta' = \{w_1, \dots, w_n\}$ is another basis, what is

$M_{\beta'}^{\beta'}(\phi)$?

Suppose $w_j = \sum_i p_{ij} v_i \quad P = (p_{ij})$

Then $v_j = \sum_i q_{ij} w_i \qquad Q = (q_{ij})$

Then $PQ = I$ so $Q = P^{-1}$.

Now $\phi(w_j) = \sum_i p_{ij} \phi(v_i)$

$= \sum_k \sum_i p_{ij} a_{ki} v_k$

$= \sum_\ell \sum_k \sum_i p_{ij} a_{ki} q_{\ell k} w_\ell$

You get $\phi(w_j) = \left(\sum_{\ell, k, i} p_{ij} a_{ki} q_{\ell k} \right) w_\ell$

$M^{\beta'}_{\beta'}(\phi)_{\ell j} = \sum_k \sum_i p_{ij} a_{ki} q_{\ell k}$

$= (QAP)_{\ell j}$

$= (P^{-1}AP)_{\ell j}$

$M^{\beta'}_{\beta'}(\phi) = P^{-1}AP.$

$= P^{-1}(M^{\beta}_{\beta}(\phi))P.$

<u>Def</u>. $A, B \in M_n(F)$ are <u>similar</u> if $A = P^{-1}BP$

for some $P \in GL_n(F)$.

Canonical forms — choose a "nice" matrix in each similarity class in $M_n(F)$

(Similarity is an equivalence relation)

Jordan canonical form.

V f.d. v.s. $/F$. $\varphi : V \to V$ linear trans.

Then V is an $F[x]$-module where $\left(\sum_{i=0}^{n} a_i x^i\right) \cdot v =$

$$= \sum_{i=0}^{n} a_i \varphi^i(v) \qquad \varphi^0 = id_V.$$

Notice V is f.d. $/F$ so, f.g. over $F[x]$. So the classification thm applies.

Since $\dim_F F[x] = \infty$, V has

no free part. So V is torsion.
In the elementary divisor form of the fund. class. thm. we get

$$V \cong F[x]/(f_1^{e_1}) \oplus \cdots \oplus F[x]/(f_s^{e_s})$$

some irreducible polys f_i.

Assume now all f_i's have degree one. So we can take each

$f_i = (x - \lambda_i)$ $\lambda \in F$.

(replace by an associate).

$$V \cong F[x]/((x-\lambda_1)^{e_1}) \oplus \cdots \oplus F[x]/((x-\lambda_s)^{e_s})$$

some $\lambda_i \in F$.

<u>Rmk.</u> We say F is algebraically <u>closed</u> if every irreducible in $F[x]$ has degree 1.

<u>Facts</u> $\mathbb{C}$ is alg. closed.

Also any field F is contained in an alg. closed field.

Assume $s=1$ so

$$V \cong F[x]\Big/\left((x-\lambda)^e\right)$$

let $I = \left((x-\lambda)^e\right).$

$\deg \underline{x}(x-\lambda)^e = e.$

So $w_1 = (x-\lambda)^{e-1} + I$

$w_2 = (x-\lambda)^{e-2} + I$

$\vdots$

$w_{e-1} = (x-\lambda) + I$

$w_e = 1 + I$

is an F-basis of $F[x]/I$.

Then

$$(x-\lambda)\cdot w_i = (x-\lambda)\cdot\left[(x-\lambda)^{e-i}+I\right]$$

$$= (x-\lambda)^{e-i+1} + I.$$

So $$(x-\lambda)\cdot w_i = \begin{cases} w_{i-1} & 2\le i\le e \\ 0 & i=1. \end{cases}$$

So

$$x\cdot w_i = \begin{cases} w_{i-1}+\lambda w_i & 2\le i\le e \\ \lambda w_1 & i=1. \end{cases}$$

(w_1 is an eigenvector for the x action on V.)

Let $\theta : V \longrightarrow F[x]/I$ be an $F[x]$-module iso.

$v_i = \theta^{-1}(w_i)$

$\beta = \{v_1, \dots, v_e\}$ is a basis of V.

and

$x \cdot v_i$

$$= \phi(v_i) = \begin{cases} v_{i-1} + \lambda v_i & 2 \le i \le e \\ \lambda v_1 & i = 1. \end{cases}$$

So

$$M^{\beta}_{\beta}(\phi) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & 0 \\ & & \lambda & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

A Jordan block

$J_{\lambda, e}$.

Now in general

$$\theta : V \cong F[x]/((x-\lambda_1)^{e_1}) \oplus \cdots \oplus F[x]/((x-\lambda_s)^{e_s})$$

Choose bases of each component as above,

write them in a list,
take the corresponding
basis β of V.
Then

$$M_{\beta}^{\beta}(\phi) = \begin{pmatrix} J_{\lambda_1, e_1} & & & 0 \\ & \boxed{} & & \\ & & \ddots & \\ 0 & & & J_{\lambda_s, e_s} \end{pmatrix}$$

Is the Jordan
canonical form of ϕ.

<u>Thm.</u> Let ϕ as above. There exists a basis β of U s.t. $M^{\beta}_{\beta}(\phi)$ is in Jordan canonical form.

Also, if β' is another basis s.t. $M^{\beta'}_{\beta'}(\phi)$ is also in Jordan form, this matrix is the same as $M^{\beta}_{\beta}(\phi)$ up to rearranging the blocks.

Ex.

Let $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = J$

$= J_{\lambda, 3}$

Then

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$$

$A \in M_n(F)$

if we find P s.t.

$P^{-1}AP = J$

is in Jordan form

$P^{-1}A^n P = J^n$

so $A^n = PJ^nP^{-1}$.

Ex. Find all matrices $A \in GL_2(\mathbb{C})$ s.t. $A^2 = I$.

= find all similarity classes containing a Jordan form of order 2.

if $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

then $J^2 \neq I$

$$J^2 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} \neq I$$

The only Jordan forms are $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

where $\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} = I$

or $\lambda_1^2 = \lambda_2^2 = 1$.

$$S = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$$

is the set of all Jordan forms with

Square $= I$ and,
now take all conjugates.

How many similarity classes are there?

Note $\begin{pmatrix}1 & \\ & -1\end{pmatrix}$ and $\begin{pmatrix}-1 & \\ & 1\end{pmatrix}$ are similar.

Actually only 3 similarity classes.