Lecture 8 1/22/2021

Today - rational canonical form.

F a field. V a f.d. v.s. $/F$

$\phi : V \to V$ linear trans $/F$.

So V is an $F[x]$-module,

where $x \cdot v = \phi(v)$.

V is a torsion $F[x]$-module.

Use invariant factor form

of fundamental thm -

$$V \cong F[x]/(f_1) \oplus F[x]/(f_2) \oplus \cdots \oplus F[x]/(f_m)$$

where $f_1 \mid f_2 \mid f_3 \cdots \mid f_m$

f_i nonzero, $\deg f_i \geq 1$.

Assume each f_i is monic -

leading coefficient $= 1$.

Assume now $m=1$

$$V \cong F[x]/(f) \qquad I=(f)$$

$$f = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$$

Choose a basis for $F[x]/(f)$

$$w_1 = 1 + I,\ w_2 = x + I,\ \ldots,\ w_n = x^{n-1} + I$$

Now

$$x \cdot w_i = x \cdot (x^{i-1} + I) = x^i + I = \begin{cases} w_{i+1} & 1 \le i \le n-1 \\ * & \end{cases}$$

Note $x \cdot w_n = x^n + I$

$$= -b_{n-1}x^{n-1} - b_{n-2}x^{n-2} + \cdots - b_1 x - b_0 + I$$

$$= -b_{n-1}(x^{n-1} + I) - b_{n-2}(x^{n-2} + I) \cdots$$

$$* \quad = -b_{n-1}w_n - b_{n-2}w_{n-1} - \cdots - b_1 w_2 - b_0 w_1$$

Now $\Theta : V \to F[x]/I$

a $F[x]$-iso

$v_i = \Theta^{-1}(w_i)$

let $\beta = \{v_1, \dots, v_n\}$ a basis of V. Then

$$\phi(v_i) = x \cdot v_i = \begin{cases} v_{i+1} & 1 \le i \le n-1 \\ -b_0 v_1 - b_1 v_2 \dots - b_{n-1} v_n & i = n \end{cases}$$

So

$$M_\beta^\beta(\phi) = \begin{pmatrix} 0 & & & & & -b_0 \\ 1 & 0 & & & & -b_1 \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 0 & -b_{n-2} \\ & & & & 1 & -b_{n-1} \end{pmatrix}$$

the companion matrix C_f of $f(x)$.

Now in general for
$$V \cong F[x]/(f_1) \oplus \cdots \oplus F[x]/(f_m)$$
we get a basis β of V s.t.

$$M_\beta^\beta(\phi) = \begin{pmatrix} \boxed{C_{f_1}} & & & 0 \\ & \boxed{C_{f_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{C_{f_m}} \end{pmatrix}$$

the <u>rational</u> <u>canonical</u> <u>form</u> of ϕ.

Thm. Let $\phi : V \to V$ a linear trans over F, V f.d. $/F$.
Let $f_1, \dots, f_m$ be the invariant factors of V as an $F[x]$-module where $x \cdot v = \phi(v)$.
Then there is a basis β of V s.t. $M_\beta^\beta(\phi)$ is in rational canonical form

$$\begin{pmatrix} \boxed{C_{f_1}} & & 0 \\ & \ddots & \\ 0 & & \boxed{C_{f_m}} \end{pmatrix},$$

If β' is another basis s.t. $M_{\beta'}^{\beta'}(\phi)$ is in rational canonical form then $M_{\beta'}^{\beta'}(\phi) = M_\beta^\beta(\phi)$.

Cor. Given $A \in M_n(F)$ there is a unique $B \in M_n(F)$ in rational canonical form s.t. B is similar to A.

Thm. Let $F \subseteq K$ F, K fields. Given $A, B \in M_n(F)$. If A is similar to B in $M_n(K)$ (there is $P \in GL_n(K)$ s.t. $P^{-1}AP = B$) then A is similar to B in $M_n(F)$.

Pf. Let C be the rat. can. form of A. Let C' " " " " of B (over F)

C, C' are still in rat. can. form over K. Since A and B are similar over K, $C = C'$.

So A and B are similar over F $\square$.

<u>Next</u>: relate forms of a matrix A to $\operatorname{charpoly}(A)$ and $\operatorname{minpoly}(A)$.

<u>Lemma</u>. Let $C_f \in M_n(F)$ for $f = x^n + b_{n-1}x^{n-1} + \dots + b_0$.

Then $\operatorname{charpoly}(C_f) = f(x)$.

<u>Pf.</u> Recall $\operatorname{charpoly}(C_f) = \det(xI - C_f)$.

$$\det \begin{pmatrix} x & 0 & \cdots & \cdots & 0 & +b_0 \\ -1 & x & & & & b_1 \\ & -1 & x & & & \vdots \\ & & -1 & \ddots & & \vdots \\ & & & \ddots & x & b_{n-2} \\ & & & & -1 & b_{n-1}+x \end{pmatrix}$$

Expand by minors in first row -

$$= x \det \begin{pmatrix} x & & & & & b_1 \\ -1 & x & & & & \vdots \\ & -1 & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & x & b_{n-2} \\ & & & & -1 & b_{n-1}+x \end{pmatrix}$$

$$+ (-1)^{n-1} b_0 \det \begin{pmatrix} -1 & x & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & x \\ & & & & -1 \end{pmatrix}$$

$$= x \operatorname{Charpoly}(C_g)$$

$$g(x) = x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1$$

$\underline{(-1)}^{n-1} b_0 \underline{(-1)}^{n-1}$

$= (\text{by induction})$

$x g + b_0$

$= f(x)$. $\square$.

Cor. Let A be a matrix $\in M_n(F)$ with rational form

$$C := \begin{pmatrix} \boxed{C_{f_1}} & & & \\ & \ddots & & \\ & & & \boxed{C_{f_m}} \end{pmatrix}$$

then charpoly(A)

$= f_1 f_2 \cdots f_m$.

Pf. $\det(xI - C)$

$$= \prod_{i=1}^{m} \det(xI - C_{f_i})$$

$$= f_1 f_2 \cdots f_m.$$

- The minimal poly.

<u>Def.</u> Given $A \in \underline{M_n(F)}$

and $g = \sum_{i=0}^{n} a_i x^i$

define $g(A) = \sum_{i=0}^{n} a_i A^i$

$(A^0 = I)$

Now $\varepsilon_A : F[x] \to M_n(F)$

$f(x) \longmapsto f(A)$

is a homomorphism of rings.

Let $I = (g) = \ker \varepsilon_A$

The unique monic such g is the minimal polynomial of A.

= the smallest monic poly that A satisfies.

Ex. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

charpoly $= (x-1)^2(x-2)$

minpoly$(A) = (x-1)(x-2)$

$$(A-I)(A-2I)$$

$$= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = 0$$